TESTING FOR EXPONENTIALITY USING A TWO-MOMENT ESTIMATOR AND A MEDIAN-CENTERED DISTANCE STATISTIC

by

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DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2006

MAJOR: EVALUATION AND RESEARCH

Approved by:

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Advisor Date

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DEDICATION

To my parents, Clare and Leonard, who instilled a love of reading and learning in their children; to my wife, Joan Marie Sabourin, whose patient encouragement accompanied her endurance of my extended years as a "professional student" and who gave unflagging moral support for this effort; and to our son, Paul, who, persevering the rigors of two educator-parents, has become a conscientious young man, navigating his own course in the arts and at university.

ACKNOWLEDGMENTS

In 1999, after several years' search for a Ph.D. program suitable for a full-time educator distant from doctorate-granting institutions, I entered the program in Educational Evaluation and Research at Wayne State University. I am grateful to the Detroit Chapter of the American Statistical Association and others responsible for the conference on statistics education given that Spring at WSU, at which I became acquainted with the Evaluation and Research program, speaking with several present or former students. Later, meeting with its director and my advisor to be, Dr. Shlomo Sawilowsky, I knew that my search had ended.

Beyond the single semester in mathematical statistics required in the Master of Arts program in mathematics at Central Michigan University, I elected the second semester and, subsequently, a course in the theory of statistical inference. Mathematical theory is not an optimal introduction to the distinct discipline of statistics, as I discovered when I began to teach introductory statistics for nursing students at Saginaw Valley State University. However, my professors at Central, Dr. Felix Famoye and Dr. Carl Lee, thorough statisticians and brilliant teachers, catalyzed my incipient interest in a field which I had avoided in earlier university studies. As a result, I was prepared to teach biostatistics at Saginaw Valley State University in Winter of 1993, when asked by Dr. John Mooningham, Chair of the Department of Mathematical Sciences, upon the unavailability of another instructor. Another member of the Central Michigan faculty I must acknowledge is Dr. Neil Christiansen of the Psychology Department. When I approached him concerning his course in multivariate statistics, a few years prior to my discovery of the EER program at WSU, he was generous enough to meet with me, suggest and even lend resources for informal, independent study, and review my work in subsequent correspondence. Among other things, Professor Christiansen motivated me to explore the relationship between analysis of variance and multiple linear regression. Were it not for my studies at CMU and deepened learning and appreciation through teaching statistics at SVSU, I would not have realized an affinity for the Ph.D. program in Evaluation and Research, the quantitative concentration in which offered an array of courses in statistics, data analysis, psychometrics, and experimental design which meshed well with my evolving interests.

I am indebted to my colleagues in the Department of Mathematical Sciences at Saginaw Valley, who offered friendly encouragement and moral support for what can be a lonely and disorienting journey, shared experiences in navigating a doctoral program, and were always willing to substitute teach for me when the need arose. Drs. Nancy Colwell and Hamza Ahmad helped find an error in the representation of one of the alternative density functions. Discussion with Dr. Steven Sepanski helped me focus a deeper analysis of product distributions, and thereby, of the mixture distributions composed of linear combinations of products of variates. Departmental colleagues and Ms. Lori Williams, faculty secretary, were forthcoming with suggestions respecting matters of style. Ms. Saun Strobel, technical secretary, was exceedingly helpful when I exhausted my skills with Microsoft Word, providing a solution for inclusion of tables which needed to be presented in landscape orientation. Colleagues in the English Department, a moiety of which shares the third floor of “Doan Science-West” with Mathematical Sciences, also provided moral support and advice respecting the protocol of doctoral studies.

While the Wayne State library resources online have been indispensable for my research, the librarians and staff at Saginaw Valley’s Zahnow Library were always friendly and obliging with answers about resources for research and were amiably indulgent of my repeated requests and inquiries, especially respecting library interloans.

Dr. John Mooningham, kindly endorsing my quest, has been extremely helpful throughout my years as a commuting, “non-traditional” graduate student in scheduling and course assignments coordinated with my need to attend classes at Wayne State and to research. Dr. Thomas Kullgren, the Dean of Science, Engineering, and Technology at Saginaw Valley during most of my time as a doctoral student, ratified the value and relevance of the EER program and was supportive with encouragement that was pivotal in affirming my pursuit of the Ph.D. and with approval of available resources that were much appreciated.

Over the years of my association with Saginaw Valley State University, others including faculty members inside and outside the College of Science, Engineering and Technology have given advice or encouragement in regards to pursuing a doctorate. The confidence in my potential and the concern that it be forged into future achievement expressed on the part of respected and admired colleagues influenced my focus on a practical objective and reinforced my dedication to this goal. Special thanks are due Dr. Leslie Whittaker, Professor of English, whose office is across the hall from my own. He has been a self-appointed catalyst to progress in my pursuit of the Ph.D. With the force of his personality and wit, he has cheered and cajoled me onward through the sometimes thorny heather of composition, his prodding punctuated with anecdotes from a vast store of experience during his variegated and adventurous life in and out of academe.

I cannot omit thanks to my physicians, Dr. William Cline, M.D., and Dr. Richard Mills, M.D., without whom I would not have survived a medically difficult year. With their skill, expertise and personal solicitude, along with the empathetic concern and support of my colleagues at SVSU, the faculty in Evaluation and Research, and my family, I was able to utilize the Autumn of 2000 most effectively in advancing towards completion of requirements in the doctoral program, while recuperating from major surgery.

Members of the administrative staff at the College of Education and at the Graduate School, Wayne State University, have been extremely helpful, promptly responding to my questions by email and phone and adeptly responding in person under the time constraints of my presence, as a distant commuter. Whenever I sought their assistance, they saved me time, travel and stress, insofar as it would have been impossible for me to visit the WSU campus from Saginaw at different times of day or on consecutive days to pursue a sequence of referrals in order to find information, secure a signature, or obtain another resource of which I had need, with time of the essence. Friendly attitudes and willingness to serve a sometimes anxious doctoral student were ubiquitous.

My professors at Wayne State, Dr. Karen Tonso and Dr. Leonard Kaplan, and EER faculty members Dr. Gail Fahoome, Dr. Donald Marcotte, and Dr. Shlomo Sawilowsky, have contributed to my growth in scholarship. The seminars guided by Professors Tonso and Kaplan, in philosophy of education and curriculum respectively, engaged lively intellectual interplay, challenged my expressive and investigative resources, and occasioned extended reading and confrontation of new ideas.

I am thankful to Dr. Marcotte for his lucid presentation of the theory behind analysis of variance, correlation and regression analysis, and the gamut of multivariate statistical methods based upon the general linear model and for his humane approach as an educator. Memories of his classes will continue to inspire a pursuit to enhance and extend my understanding of statistics and I only regret having so brief an opportunity to learn under his guidance. During my recuperation from surgery, Dr. Marcotte was particularly helpful, permitting me to minimize any interruption of my studies during this medical crisis.

Dr. Fahoome deserves special commendation for her enormous resourcefulness, dedication, and excellence in teaching a broad array of courses.

Her guidance in statistical computing, from the intricacies of SPSS to those of Structural Equation Modeling, was invaluable. She has conscientiously performed an indispensable role, teaching the essentials for graduate students' intellectual development, practical skills, and logistical knowledge as researchers. Her guidance through experimental design, research methodology, and the structure of a dissertation provided a roadmap towards success.

Dr. Sawilowsky's classes were a whirlwind of profound insight and an embodiment of intellectual controversy in the fields of robust, distribution-free, and nonparametric statistics, Monte Carlo methods, and test theory, and the time spent under his inspiration was only too brief. More than anyone else, Dr. Sawilowsky was instrumental in converting my lifelong dream, first into a feasible objective and then into an operative plan, which under his guidance never strayed far from envisioning completion, false starts and discouragements notwithstanding. He welcomed me warmly into the doctoral program and, performing outstandingly as a teacher, managed the painstaking task of directing my journey towards a doctorate, offering encouragement and sustaining counsel at those watersheds in conception, analysis, and creation and transitional nodes where focus may have tarried and motivation faltered.

I appreciate the kind efforts of the members of my doctoral – and dissertation – committee. The membership included Drs. Fahoome, Marcotte, and Sawilowsky from Evaluation and Research and Dr. Tze-Chien Sun from the Department of Mathematics at Wayne State University. I am grateful to Dr. Sun for agreeing to participate and for his kind and edifying service on my committee as an external member in my cognate area of mathematical statistics.

My sisters and parents have provided encouragement and sundry advice whenever possible, however opaque they might find the subject matter of goodness-of-fit testing. Paul Benjamin affirmatively endured competition for my time with the doctoral program and asserted his progressing maturity as an ideal son to father and mother.

Joan, my partner in life, has helped in innumerable ways, beginning with appreciating the depth of the aspiration which has propelled me. When I drove, especially homewards from Wayne State to Saginaw on dark, wintry Michigan nights, her anticipation of my safe return was a beacon of hope, not merely for the immediate trip but for the remaining expedition. Without her counsel, support, encouragement, patience, and affirmation, I could not have come this far.

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CHAPTER 1

INTRODUCTION

Testing for Exponentiality

Exponential distributions have particular theoretical importance, in part due to their elegant mathematical properties, and practical importance, due to their diverse applications in nature and in technology. Exponential distributions persist in the field of reliability analysis, from simple, constant failure rate, to mathematical models for more complex systems, with composite survival functions and with replacement policies in force (Bazovsky, 1961). A longstanding problem in applied statistics has been testing data to determine whether it originated from an exponentially distributed population, or equivalently, whether it is consistent with being a random sample of identically, independently distributed variates generated by an exponential law (Angus, 1982; Ascher, 1990; Epstein, 1958, 1960a, 1960b; Spurrier, 1984).

Monte Carlo Studies

Increasingly sophisticated tools in mathematics and statistics have been used to address this issue. In addition to asymptotic theory, an empirical approach (with simulated data as the evidentiary field), using the Monte Carlo method has evolved as an almost essential component in this pursuit. Although some research remains purely theoretical, a branch utilizing the Monte Carlo approach to present usable results, where mathematical formulae may be intractable has acquired respectability, and between these antipodes, Monte Carlo procedures have arisen alongside the power of computers to buttress the exposition of theoretical claims and to achieve lucid demonstration of their substance in practical light. Indeed, as Hoaglin and Andrews stated, “simulation plays much the same role in statistics as experimentation plays in other sciences” (1975, p. 124; and see Choi, Kim, & Song, 2004, p. 529). The majority of periodical literature cited herein reports research including a Monte Carlo study as a substantial component.

Theoretical or Empirical Questions

Approaches to testing for exponentiality supported by progressive asymptotic theory often reach a hiatus between closed formulas and practical application to data in hand. Moreover, asymptotic theory retains questionable relevance for smaller samples. Monte Carlo methods and present day computing power does permit investigation of testing approaches which span a range of mathematical complexity well beyond earlier test procedures in the vein of Kolmogorov-Smirnov or Anderson-Darling, permitting the investigator a broad selection of explicit sampling conditions. Thus, the Lorenz and Gini statistics have been adapted, from econometrics (Gail & Gastwirth, 1978a; 1978b; and Gail & Ware, 1978); and the Kullback-Liebler measure of divergence between hypothesized and empirical distributions, also referred to as relative entropy or cross-entropy (Lind, 1994, 1997; Fang & Rajasekera, 1995), has been adapted from information theory. The concept of distance to measure the separation of statistical densities has been developed in the context of differential geometry, introducing the idea of statistical curvature and employing an information theoretic metric (Amari, 1982; Critchley, Marriott, & Salmon, 1994; and see Amari, Barndorff-Nielsen, Kass, Lauritzen, & Rao, 1987). Such approaches frequently venture into branches of pure mathematics and partake of elements of the Bayesian approach to statistics (see, for example, Haughton, 1988).

A primary goal of this study is to survey that region with an emphasis on Monte Carlo simulation and experimentation, selecting from the variety of methods for testing exponentiality and from a literature spanning well over half a century, to evaluate the comparative usefulness of a relatively lucid statistical approach, without a history of theoretical, distributional analysis. A new test for exponentiality will be investigated, using the sample mean and sample standard deviation to optimize the exponential parameter, under a null hypothesis of one-parameter exponentiality, applied to simulated data, in a classical Monte Carlo study.

Questions of Application and Scope

Insofar as this is a methodological study, rather than a study involving data from phenomena, measured in the physical or social environment, the simulated variables themselves, in their statistical attributes, comprise the field of research. Similarly, the limitations of the study, in the frame of extent of applicability, will partake of the circumstances selected for the simulation of variables and the conditions selected for comparison. No data have been collected from the sensate world. The research hypotheses are, of necessity, limited and other features, such as distributions sampled synthetically, sample sizes, numbers of repetitions, are selective, in order to achieve closure of a study of this character. The researcher's hope is that such selections have been made with sufficient insight and providence to enable reasonably useful conclusions to be gleaned, on the part of readers who have partaken in such selection.

Asymptotic theory has not been engaged beyond the degree to which the project has been informed by the prior literature. A range of small and moderate sample sizes, for n as large as 100, has been considered. The methods chosen for comparison depended upon tractability to investigation and relative importance of discoveries which might emerge or relative interest in view of further research.

The primary question entailed investigation of the effects of sample size and distribution on the robustness of approaches studied in respect to type one error rate and, in some cases, power of the test. This question has been examined with respect to several different shapes of distribution or distribution functions, varying traits such as skewness, kurtosis, and other graphical or mathematical features.

Research Hypotheses

The plenary null hypothesis is that simulated data has an exponential distribution, whether specified or estimated. The experimental hypothesis arises at an antecedent level, involving the performance of a goodness-of-fit testing procedure devised and put forth here by the author. Thus, the data being artificial, its purpose – and by extension, that of the plenary null hypothesis – resides in providing an experimental environment for examining the performance of the author's goodness-of-fit statistic.

Variables

Qualification of variables.

No data have been collected from the real environment, nor was any culled from sources of data collected from the real environment. Instead, computer simulation will be used, the software including programming by the author in FORTRAN and utilities provided in or ancillary to the programming language as well as a random density generator, RANGEN (Blair, 1987), furnished by the College of Education, Department of Educational Evaluation and Research. In addition, other software, especially MINITAB, Excel, and DATAPLOT, has been employed by the author for its particular virtues in generating random data, exploratory data analysis, and mathematical manipulation to contribute to the investigation. Thus, the variables defined by the data are the parameters of this methodological study, with no actual measurements of the physical or social world, although such measurements may be considered analogous to data models studied.

Specification of variables.

One variable encompasses the distributions sampled through simulations. Some data sets were constructed artificially with particular features, such as outliers, or by mixing models. Operationally, inasmuch as the data have been simulated, some conventions on numerical range were defined, a specification which in some cases was accomplished indirectly by parameter selection.

Another procedural variable was the inferential procedure or test of hypotheses employed in the study, for Monte Carlo simulations and comparative analysis. Five tests, in addition to a new goodness-of-fit test, will be applied.

Effect size itself will be an implicit variable, incurred through simulation of various alternative distributions differing in their attributes from the null distribution, in charting a comparative analysis of statistical procedures.

Finally, to return to the use of simulated data, the variables represented in the data will consist of pseudo-random or patterned data generated by computer, and not explicitly or determinatively related to specific constructs measured in the sentient world. A caveat must be noted in regard to the use of random number generators: samples so obtained are imperfect in their quality of randomness, as they are not completely independent (see Bedford & Cooke, 2001, p. 328, footnote). The definition of these variables is purely formal, in terms of whether they may reflect a categorical, group identification, a discrete, or a continuous measure, for purposes of methodological investigation. In addition, considerations of variance will be made, thus characterizing variables by formal role in data analysis. Although definition of variables reflected in the data used will evoke neither the context of actual, experimental schema nor the subject matter of any scientific discipline, hypothetical identification of such kind, in the course of exploring different designs or distributions, may be made for the purpose of clarification or illustration.

Definition of the Monte Carlo Approach

One matter which will bear definition is the realm of Monte Carlo methods. These may be defined as the use of repeated random sampling to simulate sample spaces with theoretical or practical support. Monte Carlo methods may be operationally defined in algorithmic form and have gained an acknowledged role in statistical research (Besag & Diggle, 1977; Kromrey & Foster-Johnson, 1999; Mehta & Senchaudhuri, 2000; Mooney, 1997; Robert & Casella, 1999). Gentle (2003) described the genre of the present study, writing that “a simulation study that incorporates a random component is an experiment. The principles of statistical design and analysis apply just as much to a Monte Carlo study as they do to any other scientific experiment” (p. 298). Kleijnen stressed that “simulation implies experimentation”, specifically, “with a model” of a “real world object”, rather than with such object itself. “Stochastic simulation … also known as Monte Carlo simulation” involves experimentation with an abstract model over time, "sampling … values of stochastic variables from their distributions," using random numbers (Kleijnen, 1974, pp. 12, 14).

The designation, "Monte Carlo method", stated Shreider, first appeared in a 1949 article by that title, by N. Metropolis and S. Ulam (Buslenko, Golenko, Shreider, Sobol, & Sragovich, 1966, p. ix; and see Metropolis & Ulam, 1949). Buslenko et al. provided an extensive reference list for the early usage of mathematical simulations on electronic computers observed in their work. However, the Monte Carlo method itself was effectively used much earlier, as the authors noted, citing "Buffon's celebrated problem of needle-tossing" used to estimate pi, described by Buffon in a 1777 treatise (Buslenko et al., 1966, p. 4).

One of the topics Buslenko et al. treated is the inherent computational error in computer simulations, a matter of enduring theoretical concern, along with the reliability of simulating randomness itself (Gentle, 2003). The extent to which computing speed may have been an obstacle in early Monte Carlo efforts, however, has been reduced by the immense advance in computing power over the past half century. More directly pertinent to the current study, the authors presented the use of the uniform continuous distribution on the unit interval, along with the inverse cumulative distribution function of a probability distribution, to simulate random sampling from that distribution (Buslenko et al., 1966, pp. 314-316). This procedure of generating a random sample from a desired distribution (actually, a pseudo-random sample, given that the random sample from the uniform distribution is electronically simulated deterministically) is known as the inverse transform method (Rubinstein, 1981) or the inverse cdf method (Gentle, 2003).

Bedford and Cooke provided a concise proof. For a continuous, invertible cdf, F and U, the uniform cdf on [0, 1], for real t, and generated rv, $X = F^{-1}(U)$:

$$P(X \le t) = P\left(F^{-1}(U) \le t\right) = P\left(U \le F(t)\right) = F(t)$$

Moreover, the principle holds when F is not continuous or invertible, if $F^{-1}(u)$ is defined as $\inf\{t \mid F(t) \ge u\}$ (Bedford & Cooke, 2001, p. 27. See, also, Kleijnen, 1974, p. 26).

Assumptions concerning data

Unlike empirical research, the current study does not entail the receipt of imperfect or unknowable constructs and conditions, including populations, distributions and parameters, necessarily associated with data that relates to the physical world. On the contrary, it will be such constructs and conditions which, as if they constituted the underlying reality of a world which is knowable (at least in a probabilistic sense), will be manipulated and controlled in order to gauge the reliability of statistical procedures. Still, not every issue can be dealt with dispositively. The validity of published tools for simulation, including library functions and random generators, will be assumed. The relevance of certain probability distributions, including for instance, parametric distributions with infinite support, such as the normal, exponential, and Chi-square families, with infinite tails in at least one direction, as sometimes effective models in data analysis will be assumed. The substitution of such conveniently defined distributions for unwieldy empirical distributions for the purposes of data analysis, whether that which is conducted with real-world data, or as here, with artificial data with the objective of evaluating procedures used in the first case, is assumed at the threshold to be an intellectually and scientifically profitable enterprise.

However – and beyond the two mixtures of exponentials and normal distributions – two distributions of “messy data”, actually linear combinations of pseudo-random data from Beta and Triangular probability distributions, constructed utilizing MINITAB and Excel, are also included and treated as finite populations (N = 1500). The aim in including these is to extend the analysis to anomalously shaped data distributions which may reflect the qualities and features of data obtained from the observable world with greater verisimilitude than typically holds for the familiar, mathematically elegant distributions (see Bradley, 1968).

Limitations

Necessarily, only a limited number of configurations of the procedural variables can be constructed and employed in the investigation. If some novel insights can be culled from this experience, although they may be strictly restricted to the configurations employed, it is hoped that they may suggest broader application, which could lead to further studies.

In addition, it must be recognized that, while theory informs parts of this study, this is not a theoretical investigation in the nature of mathematical proof of general propositions or statistical properties. Rather, we depend upon the power of repetitive, pseudo-random processes, automated for electronic computer, to generate observable patterns and trends in order to explore or confirm the utility of statistical procedures.

Rationale

The distributions which shall be simulated or hypothesized, including some of those typically utilized in parametric inferential techniques, are in many cases convenient ideals but imperfect models for real-world data or the empirical populations sampled in data analysis. The abstract beauty and mathematical elegance of such procedures may not support their reliability, when applied to data which does not conform to the elegant Gaussian model, for example, which is frequently found in asymptotic theory. It is important, therefore, to understand how well or poorly conclusions dependent upon normal theory reflect the characteristics of populations or, particularly in the current study, small samples.

If one may extend Campbell and Stanley (1963) to the collection of data from the synthetic, mechanistic domain of pure procedure, one may find some comfort in their words:

> Science, like other knowledge processes, involves the proposing of theories, hypotheses, models, etc. and the acceptance or rejection of these on the basis of some external criteria. Experimentation belongs to this second phase, to the pruning, rejecting, editing phase. We may assume an ecology for our science in which the number of potential positive hypotheses very greatly exceeds the number of hypotheses that will in the long run prove to be compatible with our observations. The task of theory-testing data collection is therefore predominantly one of rejecting inadequate hypotheses. (p. 35)

Contributing to an understanding of the robustness of so ubiquitous and invaluable a procedure as goodness-of-fit testing is more than adequate rationale for the present research.

CHAPTER 2

REVIEW OF THE LITERATURE

Introduction and Fundamentals

The Exponential Distribution

The context of the exponential among continuous probability distributions.

The exponential distribution is typically treated in introductory textbooks in mathematical or calculus-based statistics. Although notation for the single-parameter exponential distribution varies, sometimes between different contexts in the same textbook, the following form, which, if not prevalent, is at least as common as any alternative, will be used for its probability density function (pdf):

$$f(x) = \begin{cases} \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases} ; \tag{2.1}$$

and refer to θ as the parameter of the distribution. (See, for example, Hogg & Tanis, 2001, p. 178.) This distribution is linked to a larger family of continuous distributions, including the gamma and Weibull, and is also connected to the discrete, Poisson distribution. (Poisson processes themselves can be viewed as members of a broader class of point processes. See Cox & Isham, 1980.)

The exponential is a special case of the gamma distribution, with the first of the latter's two parameters, α , set equal to unity. The gamma distribution has the pdf,

$$g(x;\alpha,\beta)=\begin{cases}\dfrac{x^{\alpha-1}\exp\left(-\dfrac{x}{\beta}\right)}{\beta^{\alpha}\,\Gamma(\alpha)} & \textit{for } x>0 \\ 0 & \textit{otherwise}\end{cases} \qquad \textit{where } \alpha>0 \textit{ and } \beta>0. \tag{2.2}$$

The gamma function is given by: $\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy$ for $\alpha>0$, with $\Gamma(1)=1$,

$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, and $\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)$ generally. The mean and variance of the gamma distribution are $\alpha\beta$ and $\alpha\beta^2$, respectively (Miller & Miller, 1999, p. 205, 212; Gibbons & Chakraborti, 2003, p. 15). The parameters, α and β , serve respectively as shape and scale parameters (Bury, 1999, p. 209).

The exponential distribution is also a special case of the Weibull distribution, with the second of the latter's two parameters set equal to unity. One of several forms in which the Weibull pdf is given is:

$$f(x;\alpha,\beta)=\begin{cases}\alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right) & \textit{for } x>0 \\ 0 & \textit{otherwise}\end{cases} \qquad \textit{where } \alpha>0 \textit{ and } \beta>0. \tag{2.3a}$$

The mean and variance of the Weibull distribution are $\alpha^{-\frac{1}{\beta}}\Gamma\left(1+\dfrac{1}{\beta}\right)$ and

$\alpha^{-\frac{2}{\beta}}\left[\Gamma\left(1+\dfrac{2}{\beta}\right) - \Gamma^2\left(1+\dfrac{1}{\beta}\right)\right]$, respectively (Miller & Miller, 1999, p. 213; Gibbons & Chakraborti, 2003, p. 15). The parameters, α and β , serve respectively as scale and shape parameters (Bury, 1999, p. 312,). Bury gives a common alternative form for the Weibull pdf, in which the scale parameter, σ is equivalent to $\alpha^{-\frac{1}{\beta}}$ in equation 2.3a:

$$f(x;\sigma,\lambda)=\begin{cases}\dfrac{\lambda}{\sigma}\left(\dfrac{x}{\sigma}\right)^{\lambda-1}\exp\left[-\left(\dfrac{x}{\sigma}\right)^{\lambda}\right] & \mathit{for}\ x\geq 0 \\ 0 & \mathit{otherwise}\end{cases}\quad \mathit{where}\ \sigma>0\ \mathit{and}\ \lambda>0. \qquad \text{(2.3b)}$$

With this notation, the mean and variance of the distribution are $\sigma\Gamma\left(1+\frac{1}{\lambda}\right)$ and $\sigma^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$, respectively. As the limiting form of the distribution of the smallest order statistic, for a sample from a population with bounded (lower) tail, the Weibull distribution is known as a type III extreme value distribution of minima (Bury, 1999, pp. 311-313 and see pp.267-268.) In their work on reliability engineering, Tobias and Trindade gave a form for the Weibull equivalent to Bury's. Tobias and Trindade denominated the scale parameter, σ in equation 2.3b, the 'characteristic life' (1986, p. 64; 1995, p. 82). Rohatgi, who gave a third form of the pdf combining features of equations 2.3, noted that the Weibull distribution is one which may apply in life testing, where a component deteriorates under stress, so that the rate of failure is not constant but follows an exponential law. Rohatgi observed a close relationship between the Weibull and exponential distributions:

$$\int_t^{\infty} \frac{\alpha}{\beta} s^{\alpha-1}\exp\left(-\frac{s^{\alpha}}{\beta}\right)ds = \int_{t^{\alpha}}^{\infty}\frac{1}{\beta}\exp\left(-\frac{y}{\beta}\right)dy$$, where $y=s^{\alpha}$ so that,

if T_1 has a Weibull distribution and T_2 has an exponential distribution with mean = β, then $P(T_1 > t) = P(T_2 > t^{\alpha})$ (Rohatgi, 2003, pp.406-408).

Zelen and Dannemiller offered a practical profile of this distribution:

> The Weibull distribution belongs to a class of distributions which characterize "wear out" failure, e.g. [*sic*], the longer the item has been used, the greater the probability of failure. It has been successfully applied to studies of the reliability of vacuum tubes [citation omitted]. Further, it is difficult to distinguish between it and the exponential distribution for small sample sizes (1961, p. 30).

See, also, Barlow and Proschan (1965).

Applications of the exponential distribution.

An exponential random variable may be viewed as the waiting time until the first success or occurrence of the same event, the frequency of which within an interval of time or space has a Poisson distribution. See Miller and Miller (1999, pp. 206-207), Hogg and Tanis (2001, p. 178), and Rohatgi (2003, p. 405) giving the time spent by customers at a fast food restaurant as an example of a random variable approximated by an exponential distribution. Paulson mentioned the appropriateness of the exponential distribution for "problems involving the intervals of time between events which tend to be random, as for example the interval between consecutive telephone calls, or the interval between consecutive accidents to the same worker" (1941, p. 301). Epstein stated:

> If … we view the sequence of failures and hence associated total lives as being generated by a stochastic process, then the assumption that total lives between successive failures are drawn independently and at random from an exponential pdf with constant mean life θ is equivalent to

assuming that the observations are being generated by a Poisson process with constant rate $\lambda = \frac{1}{\theta}$ (1960a, p. 90).

Jowett advanced the use of an experiential, non-uniform, ‘exposure’ scale, in order to apply the exponential distribution even in the case of varying expected rate of occurrence per unit distance or time, for which case the random variable is measured in units of exposure to risk, rather than uniform units of distance or time (1958, pp. 91, 94).

Sarhan, Greenberg and Ogawa noted the utility of the one-parameter exponential distribution in modeling age at death among infants dying under one year of age (1963, p. 114). Feigl and Zelen (1965) discussed the exponential distribution of survival time, measured as the duration from either diagnosis or initiation of treatment for a disease, such as chronic leukemia, to death, in which case the single parameter is dependent on variables indicating the severity of a particular patient’s disease.

Application and properties of the exponential distribution in life testing.

An important practical application of waiting time distributions is in reliability engineering, where the random variable is the survival time, or wait time to failure, of an industrial product or physical system (Bury 1999; Elishakoff, 1999; Epstein & Sobel, 1953, 1955; E. E. Lewis, 1987; Tobias & Trindade, 1986; Tobias & Trindade, 1995; Zelen & Dannemiller, 1961). Another continuous distribution used in such cases is the Weibull distribution, which like the Gamma distribution, reduces to an exponential distribution (Petruccelli, Nandram, & Chen, 1999, p. 189).

The exponential distribution arises from two perspectives. First, the assumption of an underlying Poisson process with stochastically distributed peaks representing failure or the condition for failure leads to an exponential distribution of time to failure. On the other hand, focus upon a constant conditional rate of failure, or hazard rate – in the terminology of reliability engineering – leads to exponentially distributed time to failure (Epstein, 1958; Tobias & Trindade, 1986, p. 38; Tobias & Trindade, 1995, pp. 48-49). In the context of life testing, an exponential distribution with parameter θ (or in the two-parameter case, with scale parameter θ) reflects a mean time between failures of θ (Epstein, 1958, p. 5; Spurrier, 1984, p. 1636; Tobias & Trindade, 1986, p. 41; Tobias & Trindade, 1995, pp. 51-52;see also Jowett, 1958). The distributions of time to failure and of time between failures (or "interarrival times"; see Ross, 1970) are the same, as a result of the so-called "lack of memory", or 'memoryless' property of the exponential distribution (Tobias & Trindade, 1986, p. 43; Tobias & Trindade, 1995, pp. 53-54; see also Epstein 1960a). Analytically, a random variable, X, is said to be without memory, or memoryless, if $P\left(X > s+t \mid X > t\right) = P\left(X > s\right)$ for all nonnegative s and t (Ross, 1970, p. 9). The exponential distribution can be described as "corresponding to a purely random failure pattern," with the meaning that a Poisson process underlies whatever causes the occurrences of failure, such as the blips from a Geiger counter actuated by a radioactive source with constant rate of emission (Epstein, 1958, p.

4; see also Jowett, 1958). Bury explains the relevance of the memoryless property, which:

> implies that the conditional probability of an event, given some related information, equals the unconditional probability of that event … . Thus, past events do not influence the probability of future events. In the context of reliability engineering, this means that an Exponentially distributed component with an operating history is as good as a new component, implying that there is *no wear out* through use (Bury, 1999, p. 181, emphasis in original).

Another property of Poisson processes giving rise to an exponential distribution of waiting, or arrival times, and of interarrival times pertains to the distribution of the waiting time until the nth event. If X_i represents the sequence of interarrival times, distributed exponentially with mean $\theta = \frac{1}{\lambda}$, let $S_n = \sum_{i=1}^{n} X_i$, yielding the sequence of waiting times. Then S_n is a random variable with the gamma distribution with parameters n and λ, and pdf given by

$f_{S_n}(x) = \lambda e^{-\lambda x} \cdot \frac{(\lambda x)^{n-1}}{(n-1)!}$, for $x \geq 0$ (Ross, 1970, pp. 16-17).

In spite of the importance of the exponential distribution for the historical development of methods for reliability analysis, and of its continuing application as “more of an exception than the norm,” where “the constant failure rate assumption can be justified,” it is necessary to take note of the contemporary view in reliability engineering, as expressed in a publication by the producer of software for reliability analysis using the Weibull and other distributions. The exponential distribution is “inappropriate for most ‘good’ reliability analyses because it does not apply to most real world applications" for the reason, most simply put, that "most items in this world do wear out" (Reliasoft Corporation, 2001). For an interesting discussion of the comparative merits of the Weibull, gamma, and generalized exponential distributions, See Gupta and Kundu (1999).

Watson and Leadbetter (1964) explained this well much earlier in the development of reliability theory. If F(t) is the cumulative distribution function of the time to failure, t, with density function, f(t), then 1 – F(x) gives the survival distribution at time, t, and $h(t)=\dfrac{f(t)}{1-F(t)}$, the hazard function, gives the conditional failure rate, the rate of failure for a unit which has survived to time, t. For all other failure-time distributions than the exponential, the hazard function varies: "engineers typically envisage h(t) as high initially (t small) due to gross manufacturing mistakes[2]. Then may follow a period in which h(t) is fairly constant, i.e., a period in which failures occur 'at random'. Then the hazard curve rises as the article 'wears out'" (1964, p. 175).

In tests for exponentiality.

Two common hypotheses pertain respectively to a one parameter and to a two parameter exponential distribution. In the first case, it is hypothesized that a set of variables represented by a given sample share the pdf in equation 2.1 for some unspecified parameter $\theta > 0$. In the second case the hypothesis states

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[2] The time variable has been changed here from x, in the original, to t.

that the variables share the pdf

$$f\left(x;\,\theta,\,A\right)=\begin{cases}\dfrac{1}{\theta}\exp\left[\;-\;\dfrac{\left(x-A\right)}{\theta}\right], & for\;\; x\geq A\\ 0, & otherwise\end{cases} \tag{2.4}$$

for some unspecified scale and location parameters, $\theta > 0$ and $\gamma \geq 0$ respectively (Bury, 1999; 178; Epstein & Sobel, 1954; Spurrier, 1984, pp. 1635-1636).

The Empirical Distribution

The empirical distribution of a finite, random sample from a continuous distribution reflects the data as order statistics upon a hypothetically continuous scale variable from a sampled probability distribution which may be known or unknown. The empirical distribution is important for the Monte Carlo method. The cumulative distribution function (cdf) of the sample data may be seen as an estimate of the cdf of the sampled population (Hollander & Wolfe, 1973, p. 446; Gibbons, 1985, p. 73; Gibbons & Chakraborti, 2003). Hollander and Wolfe defined the empirical cdf for a random sample, size m, as

$$F_m\left(a\right)=\frac{\text{the number of X's in the sample that are less than or equal to a}}{\text{m}}$$

Gibbons and Chakraborti presented the formal structure of the empirical distribution, leading to asymptotic theory of the empirical cdf. Although asymptotic theory is not the topic herein, this exposition of the empirical cdf is of interest here. Gibbons and Chakraborti (2003) define this cdf, $S_n(x)$, for any real number, x: $S_n(x)$ is the proportion of sample values (consistent with Hollander et Wolfe, *supra*) which do not exceed x:

> $S_n(x)$ is a step (or a jump) function, with jumps occurring at the (distinct) ordered sample values, where the height of each jump is equal to the reciprocal of the sample size (p. 37).

Letting $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample, its empirical distribution function is defined symbolically as:

$$S_n(x) = \begin{cases} 0 & if\ x < X_{(1)} \\ \frac{i}{n} & if\ X_{(i)} \le x < X_{(i+1)} \quad for\ i = 1, 2, \ldots, n-1 \\ 1 & if\ x \ge X_{(n)} \end{cases} \qquad (2.5)$$

> (Gibbons & Chakraborti, 2003, p. 37 [error in subscripts corrected]; see also Feller 1971; Rohatgi, 2003, p. 661).

Gibbons and Chakraborti proceeded to define the empirical cdf, $S_n(x)$, itself as a random variable related to the cdf of the underlying population from which the sample was taken: “Let $T_n(x) = n S_n(x)$, so that $T_n(x)$ represents the total number of sample values that are less than or equal to the specified value x.” Then “[f]or any fixed real value x, the random variable $T_n(x)$ has a binomial distribution with parameters n and $F_X(x)$,” where $F_X(x)$ is the unknown cdf of the population from which the sample of size n was drawn (Gibbons & Chakraborti, 2003, pp. 37-39, Theorem 3.1). The foregoing result has the following

Corollary 3.1.1 The mean and the variance of $S_n(x)$ are

a) $E\left[S_n(x)\right] = F_X(x)$ (2.6)

b) $Var\left[S_n(x)\right] = F_X(x)\left[1 - F_X(x)\right]/n$" (2.7)

(Gibbons & Chakraborti, 2003, pp. 38-39).

Gibbons and Chakraborti advanced to the asymptotic qualities of $S_n(x)$ as an estimator of the true population distribution:

> Part (a) of the corollary shows that $S_n(x)$, the proportion of sample values less than or equal to the specified value of x, is an *unbiased* estimator of $F_X(x)$. Part (b) shows that the variance of $S_n(x)$ tends to zero as n tends to infinity. Thus, using Chebyshev's inequality, we can show that [for any fixed x] $S_n(x)$ is a consistent estimator of $F_X(x)$ … or, in other words, $S_n(x)$ converges to $F_X(x)$ in probability (Gibbons & Chakraborti, 2003, p. 39, including Corollary 3.1.2).

Convergence in probability is a form of the weak law of large numbers, while convergence with probability one is a form of the strong law of large numbers (Hogg & Craig, 1978, p. 188; and see Feller, 1971). A result consistent with the strong law, the Glivenko-Cantelli Theorem, also holds:

$S_n(x)$ converges uniformly to $F_X(x)$ with probability 1, that is

$P\left[\lim_{n\to\infty} \sup_{-\infty<x<\infty} \left|S_n(x) - F_X(x)\right| = 0\right] = 1$ (Gibbons & Chakraborti, 2003, p. 39;

see also Gnedenko, 1966, p. 391; Rohatgi, 2003, p. 277).

An additional, asymptotic property of the empirical cdf is that, when standardized with respect to the true population distribution – which, as the sampling distribution of the empirical distribution, yields the mean and standard deviation (Equations 2.6 and 2.7 respectively) – the empirical distribution approaches the standard normal distribution:

$$\lim_{n\to\infty} P\left\{ \frac{\sqrt{n}\left[S_n(x) - F_X(x)\right]}{\sqrt{F_X(x)\left[1 - F_X(x)\right]}} \le t \right\} = \Phi(t),$$

where $\Phi(t)$ is the standard normal cdf (Gibbons & Chakraborti, 2003, p. 39; see also Rohatgi, 2003, p. 661.)

General Considerations: Four factors to consider in testing (Spurrier)

Spurrier (1984, pp.1636-1637) lists four factors or issues in choosing a test for exponentiality. The first issue is whether a one- or two-parameter distribution is to be tested. (See above.)

The second issue involves the type of alternatives to which the test should be sensitive. An omnibus test is one which detects all departures from exponentiality. In lieu of an omnibus test, a test may be sensitive to members of a class of distributions but not address other alternatives. Lastly, the alternative of interest may be very specific such as the example given by Spurrier, Weibull distributions with shape parameter greater than unity.

The third issue is whether and if so, what type of censoring has occurred.

In life testing, an uncensored or complete sample consists of the exact times of failure for all units in the sample. Typically, it may be impractical to collect a complete sample. Type I censoring occurs in an experiment in which a given number, n of units are tested for a predetermined, fixed time, in which some number of units, r < n fail and n – r units survive. The failure times for the surviving units are right-censored (that is, the unknown failure times are larger than the known failure times). Type II censoring occurs when the testing ceases after a predetermined, fixed number, r of failures, rather than after a fixed duration. Again, the failure times for the surviving n – r units are right-censored. Bury refers to a sample subject to Type I or Type II censoring, respectively, as a time-censored or failure-censored sample. In the first case, the number of failures before censoring is a random variable, while in the second, the censoring time (time of the last failure) is random (Bury, 1999, p. 180).

Censoring also occurs when exact failure times are not observed, in the cases of ‘readout’ or interval data. Multicensoring occurs when, due to planned or unplanned experimental conditions, still-operating units are removed from testing or censoring of different types coexists, or different failure modes are combined (Bury, 1999, pp. 179-180; Tobias & Trindade, pp. 31-34; see also Lewis, 1987). It is important to note that censoring may be intentionally incorporated into testing, in order to complete testing within a duration consistent with manufacturing needs, to isolate the mode of failure under study, or to address a particular portion of the failure rate curve (Lewis, 1987, pp. 130-131). The problems involved in maximum likelihood estimation based on grouped, truncated, or censored data are noted, also, by Zacks (1971, pp. 256-257).

The fourth issue mentioned by Spurrier is the accuracy of the data. For example, statistic involving logarithmic functions require accurate measurement of data, particularly for arguments near zero.

The Literature on Testing for Exponentiality

A Chronological Review

Before 1960.

Paulson (1941) investigated power and bias in several likelihood-ratio tests. The power functions were derived, in closed mathematical form, for: 1) a test of the hypothesis that a single sample comes from an exponential population with a given location parameter, with scale parameter assumed known equal to unity; 2) a test as in 1) but with unknown scale parameter; and 3) a test of the hypothesis that two samples come from exponential distributions with the same location parameter, assuming equal scale parameters for the two populations. All three tests were found to be unbiased on theoretical grounds.

Relevant to goodness-of-fit testing, in 1952, Anderson and Darling published theoretical results pertaining to their development of the Cramer-von Mises and Kolmogorov-Smirnov statistics by means of a weight function “to be chosen by the statistician so as to weight the deviations according to the importance attached to various portions of the distribution function," a choice which "depends on the power against the alternative distributions considered most important" (Anderson & Darling, 1952, p. 194). In a later article, the authors presented significance points under the asymptotic distribution for their test with a particular weight function that is sensitive to difference in the tails between the true and hypothesized distributions (Anderson & Darling, 1954). The weight function thus adopted, for what has become known as the Anderson-Darling (A-D) statistic, is $\psi(u)=\frac{1}{u(1-u)}$, within the context of a CvM-type test, the reciprocal of the standard deviation of an empirically distributed variate under the hypothesized cdf $F(x)$, yielding the A-D statistic,

$$W_n^2 = n\int_{-\infty}^{\infty}\left[F_n(x)-F(x)\right]^2 \psi\left(F(x)\right)dF(x),$$

where $F_n(x)$ is the empirical distribution (1954, p. 767). Further discussion of the methods of goodness-of-fit testing initiated by Cramer, Kolmogorov, Smirnov and von Mises may be found in Darling (1955; 1957).

Epstein and Sobel (1954) investigated theoretically a life test with assumed, two-parameter exponential distribution of the form in Equation 2.4. A quantity, N, of items is divided into k groups, each of which is tested with type two censoring, that is, with predetermined number, r_j, of the first failures to occur, out of the predetermined n_j elements in the jth group. Three cases are treated: 1) each of the k sets is distributed with a known location parameter, A_j;

2) all N items are distributed with a common, unknown location parameter, A; and 3) each of the k sets is distributed with an unknown location parameter, A_j. For each case ($i = 1, 2, 3$), a uniformly minimum variance, unbiased estimator, θ_i^* is derived for the scale parameter, θ, of the exponential distribution. Each θ_i^* is given as the product of a constant, C_i, and the maximum likelihood estimate, $\hat{\theta}_i$. In case two, the single, unknown location parameter, A, is estimated by $\hat{A}$, the overall minimum of the $R = \sum_{j=1}^{k} r_j$ observed failures for the N items in all k groups. In case three, the minimum observation, or first failure, X_{j1}, for each group is used to estimate the unknown A_j. The distribution of the estimators is shown to depend only on R, θ, and, in case three, also on k.

Bartholomew (1957) compared the power of three different tests for the one-parameter exponentiality of a sample, considering four different alternatives to an exponential distribution. The four alternatives are: 1) a gamma distribution (see equation 2.2 and see Bury, 1999, pp. 208ff.); 2) a distribution which the author describes as Weibull but which varies slightly in pdf from a Weibull distribution (see p. 263; equation 2.3; and Bury, 1999, p. 311); 3) a Pearson Type XI curve, $p(t) = \lambda(1 + a\lambda t)^{-\left(\frac{a+1}{a}\right)}$ (p. 253); and 4) "a special case of the Laguerre series … obtained from the Pearson Type III distribution …" (p.254), $p(t) = \lambda e^{-\lambda t}\left\{1 + a(\frac{1}{2}\lambda^2 t^2 - 2\lambda t + 1)\right\}$, $t \geq 0$ (p. 254).

The three test statistics which Bartholomew considered are:

$$M = -2\left\{\sum_{i=1}^{n} \ln t_i - n \ln \bar{t}\right\},$$

$$S = \frac{\sum_{i=1}^{n} t_i}{\left(\sum_{i=1}^{n} t_i\right)^2} \qquad \text{(p. 253).}$$

$$\varpi = \frac{\sum_{i=1}^{n} \left| t_i - \bar{t} \right|}{2n\bar{t}}, \quad where\ \bar{t} = \frac{\sum_{i=1}^{n} t_i}{n}$$

Bartholomew cited Darling (1953; and see 1962 [correction notes]) for the distribution theory respecting these criteria. Exact or approximate power functions were not available, and the author made use of asymptotic relative efficiency (ARE) (Bartholomew, 1957, p. 253; and see Gibbons, 2003). The conclusions are summarized in an ARE table. Citing results in Cox and Stuart (1955), Bartholomew defined the ARE, sometimes referred to as Pitman efficiency:

> If there are two consistent test statistics Φ_1 and Φ_2, of a hypothesis $H_0 : \theta = \theta_0$, the asymptotic relative efficiency … is the reciprocal of the ratio of sample sizes required to attain the same power against the same alternative H_1, taking the limit as the sample sizes tend to infinity and as H_1 tends to H_0 (1957, p. 254).[3]

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[3] Gibbons and Chakraborti explained with greater precision and clarity:

"Let A and B be two consistent tests of a null hypothesis H_0 and alternative hypothesis H_1, at significance level α. The *asymptotic relative efficiency* (ARE)

Respecting relative power efficiency generally, Gibbons and Chakraborti note the importance that conditions for both tests be similar and observe that “power efficiency … is difficult to calculate and interpret … [a problem which] can be avoided in many cases by defining a type of limiting power efficiency”, namely, the ARE (2003, p. 26).

The form of alternative has “a pronounced effect on the relative power of the three tests considered,” Bartholomew concluded, adding an explanation which is instructive beyond the particular tests involved:

> On alternatives I and II, M is clearly the best test to use while the very reverse is true for III and IV. The reason for this can be appreciated by noting that it is the relatively small intervals that are most important in determining whether M is significant. We should therefore expect M to be the most powerful under alternatives showing a marked departure from the exponential near t = 0. This is in fact the case for I and II which take either zero or infinite values at this point. On the other hand, the value of S, which is a sum of squares, is influenced most by the largest values and so would be sensitive to departure from the exponential at the upper tail. The ϖ test takes up an intermediate position in both cases, giving less weight to the small values than does M and more than S, and vice versa for the large values (Bartholomew, 1957, p 256).

Jowett (1958) is the earliest of the articles cited here to mention Monte Carlo methods. Jowett first derives the one-parameter exponential distribution as the asymptotic distribution of the interval between random events. Then, features

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of test A relative to test B is the limiting value of the ratio ${n_b}/{n_a}$, where n_a is the number of observations required by test A for the power of test A to equal the power of test B based on n_b observations while simultaneously of the exponential distribution are presented, including in addition to those discussed above, the skewness, γ_1, which equals 2 and the kurtosis, which equals 9 (although Jowett erroneously gives γ_2 as 6; see Sawilowsky & Fahoome, 2003, p. 348, which gives the median, as well, $\theta \ln 2$).

Jowett proceeded to discuss an heuristic routine for Monte Carlo trials used by students at the University of Sheffield, in which a model is constructed of the arrival of out-patients and time lost by doctors in waiting, at a hypothetical Hospital. Rather than a random number table, non-uniformly calibrated, circular spinners, with sector area decreasing for sectors delimited by higher scale values, are used to generate exponential data representing time intervals between patients' arrival and consultation with a doctor.

Epstein, 1960.

Epstein (1960a, 1960b), parts I and II, surveying tests for exponentiality in life testing, represented an early watershed in the field. Epstein, a seminal and major investigator of the exponential distribution (see also 1958; Epstein & Sobel, 1953, 1954, 1955; Epstein & Tsao, 1953), begins with a graphical procedure, useful for large samples. Using the transformation,

$y = \ln\left[\frac{1}{1-F(x)}\right] = \frac{x}{\theta}$, where $F(x)$ is the exponential cdf,

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$n_b \to \infty$ and $H_1 \to H_0$" (2003, p. 26).

$$F(x)=\begin{cases}1-\exp\left(-\frac{x}{\theta}\right) & \text{for } x\geq 0\\ 0 & \text{otherwise}\end{cases}, \tag{2.8}$$

and plotting y against x yields a line through the origin, with slope $\frac{1}{\theta}$. Given a random sample, x_i, $i=1,\ldots,n$, let $F(x_i)=\frac{i}{n+1}$, the expectation of $F(x_i)$ and compute y_i in accordance with the transformation above. If the sample is from the exponential distribution, then the plot of (x_i, y_i) should approximate a line through the origin. In the case that the population has the two-parameter exponential distribution (see equation 2.4), plotting y against x will yield a line which intersects the x-axis at the point $(A,0)$ rather than at the origin (Epstein, 1960a, p. 83).

Epstein proceeded to the Chi-square test for goodness of fit, using the estimate, $\hat{\theta}$ based on the data, for expected frequencies inside appropriately selected intervals. The author readily dispensed with this test, however, noting among its drawbacks, “its large sample character and dependence upon the choice of the number and position of the intervals into which the time axis is divided” (1960a, p. 85).

Next, a test based upon a property of Poisson processes is discussed. If n events occur when a Poisson process is observed for a fixed duration, T, the n times measured from time zero comprise n independent observations of a random variable which is uniformly distributed on the interval, (0, T). For “even moderately large” n, the sum of the observed times is approximately normal in distribution, with mean $\frac{nT}{2}$ and variance $\frac{nT^2}{12}$. If type II, rather than type I censoring occurs, then similar statements will hold, if n is replaced by n-1, and T by the observed time for the last, nth event. It is assumed in the foregoing that failed items, in the context of life testing, are replaced by new ones, although adjustment can be made if they are not by incorporating a “total life” function. In such a test, deviation from a constant failure rate is evidence against exponentiality. Forms of deviation include clustering of failures early in the duration and systematically changing failure rate, whereas observing too few early failures may be evidence of a two-parameter exponential distribution (Epstein, 1960a, pp. 85-86).

Next, Epstein considered tests based upon the independence, in the case of an exponential distribution, of the total lives between successive failures. These tests focus on respectively different phenomena over different portions of the range of observations. A test for an “abnormally small” first failure time utilizes an F-distributed test statistic, comparing Chi-square statistics based on the intervals $[0, \tau_1]$ and $[\tau_1, \tau_r]$, where $\tau_1 \le \tau_2 \le ... \le \tau_r$ are the first r failure times. The Chi-square statistics here are derived from the distribution of the total time on test, which for singly, right-censored samples can be given as $T = r\hat{\theta}$, where $\hat{\theta}$ is the estimate of mean time to failure for a sample of r failure times, out of a total of n items tested. Although $\hat{\theta}$ may take different forms, with right-censoring alone, the maximum likelihood, unbiased, minimum variance, efficient, and sufficient estimator is given by:

$$\theta_{r,n} = \frac{x_{1,n} + x_{2,n} + ... + x_{r,n} + (n-r)\,x_{r,n}}{r}. \tag{2.9}$$

Then $\frac{2T}{\theta} = \frac{2r\theta_{r,n}}{\theta} \sim \mathrm{X}^2_{2r}$ (Epstein & Sobel, 1953, 1955; and see Bury, 1999). For the test on τ_1, the total life in the intervals $[0, \tau_1]$ and $[\tau_1, \tau_r]$ respectively have the independent distributions, $\frac{2T_{(\tau_1)}}{\theta} \sim \mathrm{X}^2_2$ and $\frac{2T_{(\tau_r - \tau_1)}}{\theta} \sim \mathrm{X}^2_{2r-2}$, so that the test statistic, $\frac{(r-1)T_{(\tau_1)}}{T_{(\tau_r-\tau_1)}}$ is distributed as $F_{2,2r-2}$. The critical value in this test will be a "lower α" of the $F_{2,2r-2}$ distribution The test can be extended to test whether both τ_1 and τ_2 are abnormally small, in which case the test statistic, $\frac{(r-2)T_{(\tau_2)}}{2T_{(\tau_r-\tau_2)}} \sim F_{4,2r-4}$ is used with a left-tailed critical value (Epstein, 1960a, pp. 86-87).

Generally, the underlying distribution for a test which focuses on the minimum of a random sample is known as a smallest extreme value distribution. Such a distribution may be appropriate where "*many identical and independent competing processes*" lead to failure, "and the *first* to reach a critical state determines the failure time." The only life distribution among such smallest extreme value distributions is the Weibull distribution, of which the exponential distribution is a special case (italics in original, Tobias & Trindade, 1986, pp. 70-71; Tobias & Trindade, 1995, pp. 89-91).

The fifth test treated by Epstein is one for "an abnormally long first failure." Utilizing the same test statistic as in the previous test but is right-tailed, the criterion being an "upper α" of the $F_{2,2r-2}$ distribution (1960a, p. 87).

The sixth test compares the mean life, or rate of failure, in the first part of a life test with the mean in the remaining part of the test and "can detect gross changes over time." In general, the total life in the interval $[0, \tau_k]$ may be compared with the total live in the interval $[\tau_k, \tau_s]$, where k < s. The respective Chi-square statistics, $\frac{2T_{(\tau_k)}}{\theta} \sim \mathrm{X}^2_{2k}$ and $\frac{2T_{(\tau_s-\tau_k)}}{\theta} \sim \mathrm{X}^2_{2s-2k}$, lead to a test statistic $\frac{(s-k)T_{(\tau_k)}}{kT_{(\tau_s-\tau_k)}} \sim F_{2k,\,2s-2k}$, which may be employed in a one- or two-sided test, depending upon the type of change being investigated (Epstein, 1960a, p. 88).

The next test treated arranges the date into k groups of r failure times each and focuses on whether θ fluctuates from group to group, applying Bartlett's test (see Hampel, Ronchetti, Rousseeuw, & Stahel, 1986, p. 188) for homogeneity of variance. This approach is enabled by the mutual independence of the total lives in the k respective groups. For each group, the statistic

$$\frac{2rk\left\{\ln\frac{T\left(\tau_{kr}\right)}{k} \;-\; \frac{1}{k}\left[\ln T\left(\tau_{1r}\right)+\ln T\left(\tau_{2r}-\tau_{1r}\right)\;+\ldots+\;\ln T\left(\tau_{kr}-\tau_{(k-1)r}\right)\right]\right\}}{1 \;+\; \frac{k+1}{6rk}} \qquad (2.10)$$

Is distributed as Chi-square with k – 1 degrees of freedom. An extreme right-tailed value of this statistic is evidence against the null hypothesis that θ is the same over the duration of the life test (Epstein, 1960a, p. 88). This test statistic is best used when testing the distribution of total lives between successive failures, for exponentiality against a Weibull alternative (1960a, p. 90).

The eighth test is a special case of the previous test, with a single observation only in each group, in which case r = 1 and the test statistic becomes

$$\frac{2k\left\{\ln\frac{T\left(\tau_{kr}\right)}{k} \;-\; \frac{1}{k}\left[\ln T\left(\tau_{1}\right)+\ln T\left(\tau_{2}-\tau_{1}\right)\;+\ldots+\;\ln T\left(\tau_{k}-\tau_{k-1}\right)\right]\right\}}{1 \;+\; \frac{k+1}{6k}} \sim \mathrm{X}^2_{k-1} \qquad (2.11)$$

Epstein characterizes this test as “checking whether or not θ is constant from one observation to the next” and remarks that formula 2.11 is also the test statistic which would be used in a likelihood ratio test of the hypotheses,

$H_0 : f\left(x;\theta\right)=\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)$ against $H_1 : f\left(x;\theta,\beta\right)=\frac{1}{\Gamma\left(\beta\right)\theta^{\beta}}\;x^{\beta-1}\exp\left(-\frac{x}{\theta}\right)$, that

is, x has an exponential distribution versus x has a gamma distribution. This test, Epstein advises, is an alternative to the Chi-square test or to the third test given, based on uniformity of the conditional distribution of ordered total lives (Epstein, 1960a, p. 89). Epstein remarks two other distributional relationships leading to tests similar to Bartlett's test. For total life, $T(\tau_i)$ and preassigned total life $T*$ for a random r failures, because $\frac{T(\tau_i)}{T*}$ is uniformly distributed, $-2\sum_{i=1}^{r}\ln\left[\frac{T(\tau_i)}{T*}\right] \sim X_{2r}^2$. Similarly, if testing is terminated after a predetermined number, r of failures, then $-2\sum_{i=1}^{r-1}\ln\left[\frac{T(\tau_i)}{T(\tau_r)}\right] \sim X_{2r-2}^2$ (Epstein, 1960a, pp. 89-90).

The ninth test, due to Hartley and based upon the maximum F distribution, Epstein characterizes as a “quick alternative procedure to the eighth” which is “almost as efficient as the Bartlett homogeneity test” (Epstein, 1960a, pp. 90-91).

Epstein's tenth category is for “tests for abnormally long periods in which there is no failure.” These include techniques based upon separate work by R. A. Fisher and W. G. Cochran (Epstein, 1960a, pp. 91-92).

Next Epstein dealt with a standard, graphical method, the Kolmogorov-Smirnov test (1960a, pp. 92-93). The last, twelfth topic in the article is based upon the lack of memory property of the exponential distribution and the relationship between the distribution of failure times or the cdf of the life distribution, and the hazard function, as known in reliability engineering. Tobias and Trindade treat this relationship. Let $f(t)$ be the pdf of the life distribution and $F(t)$ be the cdf, or cumulative probability of failure at time t. (For the one-parameter exponential, this is $F(t) = 1 - \exp\left(-\frac{t}{\theta}\right)$.) $R(t) = 1 - F(t)$, the survival, or reliability function can be viewed as either the probability that a randomly selected unit will still be operating after time t or as the proportion of all units that will survive at least that amount of time. Then, the hazard function, or instantaneous failure rate (for which the caveat is apropos that it is not a probability) is given by $h(t) = \frac{f(t)}{R(t)} = \lim_{\triangle t \to 0}\left(\frac{F(t+\triangle t) - F(t)}{R(t)\cdot\triangle t}\right)$. The cumulative hazard function, $H(t) = \int_0^t h(t)dt = \int_0^t \frac{f(t)}{R(t)}dt = -\ln\left[R(t)\right]$, as $\frac{d}{dt}R(t) = \frac{d}{dt}\left[1 - F(t)\right] = -f(t)$. The equation, $H(t) = -\ln\left[R(t)\right]$ can be solved for the cumulative life distribution in terms of the hazard function, $h(t)$:

$$F(t) = 1 - \exp\left[-H(t)\right] = 1 - \exp\left[-\int_0^t h(y)dy\right]. \tag{2.12}$$

It follows directly that if the hazard function gives a constant failure rate, $\frac{1}{\theta}$, the life distribution is exponential. See Tobias and Trindade (1986, pp. 21-26, 1995, pp. 28-33).

Epstein's twelfth test applies the foregoing mathematical relationship to investigate the life distribution, upon comparing the ratios of numbers of failures in successive intervals. If the data is from an exponential population, these ratios “should fluctuate within reasonable limits about a constant value, namely the failure rate” (Epstein, 1960a, p. 94).

Part II (Epstein, 1960b), furnishes practical examples of the tests in part I.

The 1960s.

Relying upon work by Epstein and Sobel cited above, Zelen and Dannemiller (1961) explored the behavior including robustness of tests for exponentiality (one-parameter), as against a Weibull alternative in four cases: a complete test which continues until all components fail; fixed sample size n, with stopping after r < n components fail; truncated, non-replacement testing, placing n components simultaneously and stopping after either a preassigned time or the rth failure, whichever occurs first; and sequentially testing one item at a time, deciding whether to accept or reject or else run another test, after each failure. None of the testing procedures, studied by simulation, were robust. The authors appended treatment of a “general life distribution”, including moments and characteristic function.

Sarhan, Greenberg, and Ogawa (1963) sought the minimum variance, unbiased linear estimators for the parameters of the two-parameter and one-parameter exponential distributions, with the restriction that only two order statistics from a sample, size n, would be used. For samples, sizes $n \le 21$, the authors determined the optimal linear combination of two specific order statistics, $x_{(1)} \le x_{(l)} < x_{(m)} \le x_{(n)}$ for these estimators. For the two-parameter distribution,

$x_{(l)} = x_{(1)}$ and the index of the second of the two order statistics, $x_{(m)}$, is a value between n and n – 4, decreasing as the sample size increases, the same two order statistics serving for both estimators. The authors furnish a table with coefficients for the estimators, respectively of the location and scale parameters, for sample sizes, $n = 1, ..., 20$. Concerning efficiency of these estimators, the authors note that, in estimating the location parameter,

> the efficiency of two order statistics relative to the complete sample is high and, in fact, is never less than 93.7%, this value occurring when n = 6. The efficiency increases with sample size after this point and at n = 20 it has attained a value of 97.55%. The efficiency in estimating [the scale parameter] is not as high … and decreases almost consistently, although slowly, as the sample size increases. It approaches an asymptotic limit of 64.76% relative efficiency (Sarhan et al., 1963, pp.104-105).

It is additionally of interest that, in estimating the location parameter, "the farther l is from 1, the worse becomes the efficiency. The importance of the initial order statistic is not as great in estimating [the scale parameter], however, and the loss is not so great, particularly as n increases (Sarhan et al., 1963, p. 105).

For the one-parameter distribution, the index of the first of the two order statistics, $x_{(l)}$, is a value between n – 1 and n – 7, decreasing as the sample size increases, up to n = 21; and the index of $x_{(m)}$ is a value between n and n – 1, decreasing as sample size changes from n = 15 to n = 16. Here, "the first ordered observation is no longer crucial in the estimation procedure," as it was in the two-parameter case (Sarhan et al., 1963, p. 105). A table of coefficients is provided for the one-parameter estimator as well.

Kulldorff (1963) expanded the estimation procedure investigated by Sarhan et al. to estimating the exponential parameters “on the basis of k suitably chosen order statistics, out of a sample, size n. Coefficients, relative variance, and relative efficiency are tabulated for several values of k, for small samples. Three cases are treated for the two-parameter distribution, to wit, when either parameter is unknown and when both parameters are unknown.

Research on characterization of the exponential distribution continues as a theme of research in mathematical statistics. A finding by Ferguson may have application in testing for exponentiality, through the appropriate use of simulation for moderately sized samples. Ferguson demonstrates that, if X and Y are independent random variables with continuous pdf’s and “if $U = \min(X, Y)$ and $V = X - Y$ are independent, then both X and Y have exponential distributions with a common location parameter but with possibly different scale parameters.” The characterization follows, because the converse, “that if X and Y are independent and if each has an exponential distribution with a common location parameter but with possibly different scale parameters, then U and V are independent,” holds also (1964, p. 1200).

Bain and Weeks (1964) summarized features of the truncated exponential distribution, including the obtainability of uniformly most powerful or uniformly most powerful unbiased tests for the parameter, c, in the pdf, given as:

$$f\left(x;\, c\right) = \begin{cases} c \cdot \exp\left(-cx\right) \cdot \left(1 - \exp\left(-cx_0\right)\right)^{-1}, & for\ 0 < x < x_0 \\ 0, & otherwise \end{cases}.$$

Tanis (1964) shows that certain linear forms in the order statistics of a random sample from the two-parameter exponential distribution (see equation 2.4) are independent and have a Chi-square distribution. In addition, it is shown that the exponential distribution $f(x;\, \theta,\, A)$ is characterized by the independence of the estimators for its two parameters, to wit, X_1 for A and $\frac{\sum_{i=2}^{n}(X_i - A)}{n}$, where $X_1 < X_2 < ... < X_n$ are the order statistics. Ferguson's article was published only half a year after Tanis' article and does not cite Tanis, but the latter publication appears to offer a two-sample analogue of the earlier one.

Basu (1964), in an article published 6 months before Bain and Weeks (1964), compared the complete and truncated exponential distribution, giving type I error rates when testing a truncated population with a test for the complete distribution, tabulating lower percentile points for the truncated distribution, and investigating the power of the uniformly most powerful (UMP) test in three situations. The type I error rates are increasingly inflated from sample sizes 1-5, with percent of error growing from about 15 percent to over 100 percent.

Basu presented results for power, given for sample sizes n = 1 through n = 5, for untruncated and truncated populations, respectively, testing θ_0 = 0.5, 3, 5, 15, and 30, in the truncated case with corresponding upper endpoints, x_0 = 1, 5, 10, 25, and 50. The actual values of θ were 0.2 and 0.4 for the first set; 0.5, 1, 1.5, and 2 for the second; 1 and 3 for the third; 2 and 13 for the fourth; and 25 for the fifth, all tests being against the true, left-tailed alternative, $\theta < \theta_0$. The three power calculations for each of the 55 testing situations were 1) power of the UMP test assuming a complete exponential distribution, with an actually complete (untruncated) population; 2) power of the same test with an actually truncated population; and 3) power of the UMP test assuming a truncated exponential distribution with an actually truncated population. The calculations were performed, based on theory, using tables of the incomplete gamma function. Simulation is not mentioned in the article. A dramatic progression is seen in the table, powers of 0.8 or more being achieved for n $\geq$ 3 only in the second set, only with the greatest effect, $|\theta - \theta_0|$ = 3 - 0.5 = 2.5; in the third set, only with effect, 5 – 1 = 4; and in the fourth set, only with effect, 15 – 2 = 13. It is notable, however, that considerably high power is achieved, especially in the second and fourth sets, with sample size as small as n = 3. The highest power is achieved in the first two of the three testing situations, that is, using the UMP test for the complete exponential distribution, in which case actual truncation of the population does not have noticeable effect, whereas the UMP test based on the truncated distribution function results in somewhat lower power, for example, 0.8672 as compared to 0.9677-0.9679, when n = 4, x_0 = 0.5, θ_0 = 3, and θ = 0.5. Basu asserted that power of the usual test for a truncated variable will generally be greater than the power of the usual test for an untruncated variable, which in turn will generally be greater than the power of the test based on the truncated exponential applied to a truncated variable. Basu distinguished other researchers' work using right-tailed tests and notes that "serious errors occur both in the power and size of the tests especially when the sample size is large" and that "one must be very careful on deciding the type of truncation to be used in a particular situation" (1964, p. 213).

Observing that industrial experience in life testing indicates that the exponential distribution is often "a suitable statistical model only in a given range," Basu (1965a, p. 548; 1965b; and see Tobias & Trindade, 1986, 1995) dealt with testing for exponentiality in the presence of outliers. Citing literature of the prior three decades, Basu noted Epstein and Sobel's work with censored data and other researchers' use of mixed exponential models viewing some "early and late" failures as caused by different sources of disturbance, and then described his own, alternative approach to the problem: "[which is] to consider those "early and late" failures (especially when they constitute a small fraction of the total sample) as outliers and to develop some suitable statistical theory for the remaining observations" (Basu, 1965a, p. 548). Basu considers the appropriateness of some existing tests for this approach and proposes other tests for outliers.

After reviewing certain findings and tests for the exponential distribution due to Epstein and Sobel (1953, 1954), Basu critiqued statistics proposed by

Carlson[4] to test the hypothesis, $H_0 : \mu = \mu_0$, for the exponential lower limit, respectively $h_n = \frac{n(x_1 - \mu_0)}{\theta}$ for known and y for unknown scale parameter, θ. First, an estimator of θ based upon w_n is not minimum variance. Secondly, if x_n is “too large” due to the presence of outliers, w_n would be “a very unreliable estimate of θ,” leading to a less sensitive test. In consideration of the work of Epstein and Sobel cited, Basu proposed the following statistic, used by the former authors (1954) for confidence limits for μ: for the case where outliers are undetected “either due to non-availability of some suitable test or due to our ignorance of the situation”, which does not suffer the defects of that used by Carlson:

$$\tau^* = \frac{n(n-1)(x_1 - \mu_0)}{\sum_2^n (n-i+1)(x_i - x_{i-1})} = \frac{n(n-1)(x_1 - \mu_0)}{(x_2 + \ldots + x_n) - (n-1)x_1} . \qquad (2.13)$$

This statistic has an F distribution with 2 and (2n – 2) degrees of freedom respectively (Basu, 1965a, p. 550).

If, on the other hand, outliers are expected, say m at the lower end and $l = (n - r - m)$ at the upper end, the sample can be trimmed to size r, following

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[4] The article cited, which the researcher was unable to consult, is Carlson, P. G. (1958). Tests of hypothesis on the exponential lower limit. *Skand. Aktuartidskr., 41*, 47-54.

Tukey[5], to $x_{m+1} < x_{m+2} < ... < x_{m+r}$, to be treated as a censored sample. Basu derives the joint density of a random sample of these elements, given by

$$\phi\left(x_{m+1}, \ldots, x_{m+r}\right) =$$

$$\frac{n!}{m!(n-m-r)!}\left\{1-\exp\left[-\frac{1}{\theta}\left(x_{m+1}-\mu\right)\right]\right\}^{m} \cdot \exp\left[-\frac{1}{\theta}\left\{\sum_{1}^{r}\left(x_{m+i}-\mu\right)+\left(n-m-r\right)\left(x_{m+r}-\mu\right)\right\}\right].$$

From this joint density, Basu obtained the maximum likelihood estimates of μ and θ, respectively $\tilde{\mu} = x_{m+1}$ and $\tilde{\theta} = \frac{1}{r}\left\{\sum_{2}^{r}\left(n-m-i+1\right)x_{m+i}\right\}$[6]. Basu continued, "hence the minimum variance unbiased estimates of μ and θ are"

$\hat{\mu} = x_{m+1} - K\hat{\theta}$ and $\hat{\theta} = \frac{1}{r-1}\left\{\sum_{2}^{r}\left(n-m-i+1\right)x_{m+i}\right\}$, where $K = \sum_{1}^{m+1}\frac{1}{n-i+1}$.

Then a test of $H_0 : \theta = \theta_0$ can be performed, based upon $\frac{2\left(r-1\right)\hat{\theta}}{\theta} \sim \mathrm{X}^2_{2r-2}$ (Basu, 1965a, pp. 551-552). The hypothesis on the location parameter, $H_0 : \mu = \mu_0$, can be tested using the statistic, $\tau = \frac{x_{m+1}-\mu_0}{\sum_{2}^{r}\left(n-m-i+1\right)x_{m+i}}$, for which Basu derived

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[5] Basu cites the article, Tukey, J. W. (1962). The future of data analysis. *Ann. Math. Statist. 33*, 1-67.

[6] The notation in the article transforms from x to z for the sample variable; however, there is no explanation for this. The notation here restores the variable name, x.

the cdf,

$$F(\tau)=\frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m)}\sum_{j=0}^{m}\frac{(-1)^j\binom{m}{j}}{(n-m+j)}\cdot\left\{1-\frac{1}{\left[(n-m+j)\tau+1\right]^{r-1}}\right\}\ for\ \tau>0$$

(1965a, pp. 552-553). The author provides tables of this statistic for the 90th, 95th, and 99th percentiles and sample sizes from 2 to 20.

Basu then presented some tests for outliers, extending the literature (for which citations are omitted here), “all of which [tests] are analogues of the corresponding tests for the normal distribution” (1965a, p. 554).

P. A. W. Lewis (1965) investigated tests “useful against rather vaguely specified alternatives” to the exponential (1965, p. 67). Using computer simulation, the author compared Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling tests, with untransformed data, and with data transformed to successive differences in order statistics, and two other tests in the latter case.

Sheshadri, Csorgo, and Stephens (1969) were the first authors treated here to report experimental findings identifying the Monte Carlo method, although earlier investigators utilized repeated computer simulations with data generated to follow particular distributions. In order to test for exponentiality, the authors utilize one of two transformations of the data, the ‘J’ and ‘K’ transformations respectively, which in each case characteristically results in a uniform distribution on the interval [0, 1], under the null hypothesis that the data is exponentially distributed. The authors then explored tests for the uniform distribution which use

"Kolmogorov-type statistics", these being the Kolmogorov-Smirnov and Cramér-von Mises statistics and variations of these, and for comparison, use the Chi-square and Pearson tests for goodness-of-fit. The transformations of the observations, $y_i, i = 1,\ldots,n$ are: $z_j = J\left(y_j\right) = \dfrac{\sum_{i=1}^{j} y_i}{\sum_{i=1}^{n} y_i}$ and $z'_j = K\left(y_j\right) = \dfrac{\sum_{i=1}^{j} d_i}{\sum_{i=1}^{n} d_i}$ where $d_i = \left(n+1-i\right)\left(y_{(i)} - y_{(i-1)}\right)$ for $1 \le i \le n$, setting $y_{(0)} = 0$.

The Kolmogorov-Smirnov (S-K) goodness-of-fit test, which is treated generally in the literature, is a distribution-free test comparing, in the one-sample case, the empirical distribution function with a specified distribution function (see, for example, Gibbons & Chakraborti, 2003; Hajek & Sidak, 1967; Hollander & Wolfe, 1973; Parzen, 1962; Rohatgi, 2003). The S-K test statistic, sometimes given in analytical terms as $D_n = \sup_x \left|S_n - F_X\left(x\right)\right|$, that is, the least upper bound of the absolute deviations between the empirical cdf and the cdf under the null hypothesis, for all real x, (Gibbons & Chakraborti, 2003, p. 112), is clearly expressed in applicable terms by Rohatgi, for a two-sided test, as:

$$D_n = \max_{1 \le i \le n}\left\{\max\left[\frac{i}{n} - F_0\left(x_{(i)}\right),\ F_0\left(x_{(i)}\right) - \frac{i-1}{n}\right]\right\},$$

where F_0 is the hypothesized cdf and the $x_{(i)}$ are the ordered observations (Rohatgi, 2003, p. 756). This statistic focuses on the point where the deviation between the hypothesized and empirical distributions is greatest (Siegel, 1956, p.

48). The deviations are expected to be small under the null hypothesis, and vanish asymptotically under the Glivenko-Cantelli theorem, *supra* (and see Gibbons & Chakraborti, 2003, pp. 39, 112). The test statistic has a known, tabulated distribution for continuous distributions and, unlike the Chi-square goodness-of-fit test, applies to the data individually, not requiring grouping (Gibbons & Chakraborti, 2003; Hajek & Sidak, 1967; Hollander & Wolfe, 1973; Rohatgi, 2003). It is not practicable to perform the K-S test if the hypothesized distribution has parameters estimated from the sample, as the sampling distribution is more complicated in this case; and Lilliefors, using Monte Carlo simulation, furnished an alternative goodness-of-fit table (later improved) for this case (Gibbons & Chakraborti, 2003, p. 130 and see Lilliefors, 1969).

Using computer-generated samples from an exponential distribution with mean of unity, Jackson (1967) investigated properties of a test statistics based on order statistics, the moments and asymptotic distribution of which are analytically derived as well. Skewness and Kurtosis are also derived or given approximate form. Given a one-parameter exponential distribution, the expected value of the rth order statistic, for sample size n, is

$t_{rn} = \frac{1}{\theta} E\left(X_{(r)} \right) = \sum_{i=1}^{r} \left(n - i + 1 \right)^{-1}$. The first statistic Jackson provides is

$T_n = \frac{\sum_{r=1}^{n} t_{rn} X_{(r)}}{\sum_{r=1}^{n} X_{(r)}}$, which is “normalized to remove dependence on the nuisance parameter” of the distribution (Jackson, 1967, p. 540). The asymptotic distribution is reached slowly, and is inaccurate for n < 100, but approximations to the distribution are treated, as well as the exact cdf, which is in closed form but computationally intense. For failure- or type II-censored testing, stopped after r observations of failure, an estimate of the exponential parameter is proposed:

$\hat{\theta} = \dfrac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{r}$, where the $X_{(i)}$ are the ordered intervals between failures. The test for exponentiality then uses the statistic,

$T^* = \dfrac{\sum_{i=1}^{r} t_{rn} X_{(i)}}{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}$. Although distribution of the latter statistic is elusive, the author uses an empirical standardization and assumption of asymptotic normality to obtain a test of significance (1967, p. 547-548).

The Cramér-von Mises statistic (C-vM) was applied to the transformed data by Seshadri et al. (1969, p. 503; and see Parzen, 1962, p. 100) as

$W_n^2 = \dfrac{1}{12n} + \sum_{i=1}^{n} \left(z_{(i)} - \dfrac{2i-1}{2n} \right)^2$ and exists in alternate forms (Hajek & Sidak, 1967, pp. 92-93). The Pearson goodness-of-fit statistic, $P = -2\sum_{i=1}^{s} \ln(z_i)$, where $s = \sum_{i=1}^{n} d_i$ and the d_i are as given by Seshadri et al., as quoted above (Seshadri et at., 1969). The authors used four cells for the Chi-square test.

The authors focused on power of the test for four alternatives to the exponential distribution, the normal and Cauchy distributions, and distributions 'D1' and 'D2', given respectively by $F(y)=(1-y)^k,\ 0<y<1,\ k>0$ and $F(y)=y^k,\ 0<y<1,\ k>0$ (Seshadri et al., 1969, p. 506)[7]. Monte Carlo samples were drawn, with sample sizes 20 and 40 and the 'J' and 'K' transformations were performed, resulting in samples size 19 and 39 respectively for each of six test statistics, 1000 samples being drawn in each case. Alpha was set at ten percent throughout and power was expressed as percent of the 1000 samples in which lower tail or upper tail significance is found, depending upon the form of test.

For normally distributed data, power for the K-S and C-vM tests ranged between 23 percent and 33 percent, except for sample size n = 40 with the K transformation where K-S yielded 53 percent and C-vM 58 percent. The K transformation, using the upper tail of the test statistics and offering higher power, was superior generally to the J transformation, except for two variations, respectively, of the K-S and C-vM statistics: "…in the respective tails where the results are significant … everywhere K is more powerful than J if one takes the 'best' statistic of the set available" (Seshadri et al., 1969, p. 506). The Pearson

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[7] The authors did not comment on consistency of D1 and D2 with the criteria for a probability distribution.

test was nowhere preferable, and the Chi-square test was only compared for the J transformation, and comparable only for n = 20.

Higher power was observed in some cases with the D1 distributed data, more generally for Cauchy data and most dramatically for the D2 distributed data. With distribution parameter k = 3 and the K transformation, the D2 data resulted in rejection for 100 percent of the samples, as might be expected in comparing this increasing power function with the decreasing exponential pdf.

The authors noted that the K-S and C-vM statistics perform similarly and that it is difficult to decide between them with regard to power. Chi-square is comparable in a few cases for upper tail tests but has generally much lower power for the lower tail tests, using the J transformation. The authors do not state the parameter for the Cauchy data, positive only. The power for Cauchy data ranges from .51 to .95, among the K-S and C-vM statistics, whereas Chi-square ranges from .49 to .88, with mixed results for the two transformations. For the D1 data, the power is near .20 for the highest values used for the parameter k, but ranges from .37 to .96 among the K-S and C-vM statistics and from .33 to .85 for Chi-square otherwise. It is thus difficult to generalize about the experiment (Seshadri, 1969, p. 508).

Lilliefors (1969) confronted the inapplicability of the “commonly tabulated critical points” in using the K-S statistic for testing whether a sample comes from a continuous distribution, when certain parameters of the distribution must be estimated from the sample. When certain parameters of scale or location are estimated and the probability integral transformation is performed (see Gibbons & Chakraborti, 2003, p. 42), the resulting distribution depends upon the population sampled, and “one can construct tables for use with the Kolmogorov-Smirnov statistic for that particular distribution” (Lilliefors, 1969, 387). In an earlier paper, Lilliefors (1967) presented a table for use in testing for normality, when the mean and variance are unknown and estimated by the data. In the current article, the author presents a table for use with the K-S statistic, in testing for exponentiality, when the population mean, the parameter θ, is not specified and is estimated by the sample mean, $\bar{x}$. Critical values for the K-S statistic were obtained by Monte Carlo simulation, drawing 5000 samples for odd sample sizes from 3 to 19, 20, 25, 30, and 35, using smoothing methods to fill in the missing values under n = 19. The results compared well with exact critical values determined for n = 3. An IBM 360-40 computer at George Washington University was used. The divergence from the ordinary K-S table was considerable:

> Comparing Table 1 with the standard table for the Kolmogorov-Smirnov test …, it is seen that the critical values in Table 1 for a .05 significance level are for each value of N about the same as the critical values for a .20 significance level using the standard tables. Thus the result of using the standard table when values of the mean and standard deviation are estimated from the sample would be to obtain an extremely conservative test in the sense that the actual; significance level would be much lower than that give by the table (Lilliefors, 1969, p. 388).

Lilliefors also presented Monte Carlo calculations of power of the K-S test using his Table 1 critical values, with data respectively from the lognormal and Chi-square, df = 1, distributions, each with n = 10, 20, and 50 and significance levels .01, .05, and .10, 1000 samples being used in each case. In testing the lognormal data, for n = 50 and alpha = .10, power of .431 was obtained, but otherwise less than half that was achieved. In testing the Chi-square data, power above sixty percent was achieved only for sample size 50: .623 for level .01, .816 for level .05, and .929 for level .10 (Lilliefors, 1969, p. 389)[8].

The 1970s.

Uthoff (1970) investigated three scale and location invariant tests, one for normality versus uniformity, one for normality versus exponentiality, and one for uniformity versus exponentiality (the first of which was the subject of Monte Carlo studies of power, in the previous decade). The advantage of such tests is indicated by the author's explanation: "A test is scale and location invariant when the decision to accept or reject is not affected by multiplying each data point by the same positive constant or by adding the same constant to each data point" (1970, p. 1597). The author sketched "a more simple and elegant derivation" of statistics for the most powerful invariant tests, to wit, $\frac{x_{(n)} - x_{(1)}}{s} < c$, for testing a null hypothesis of normality against a uniform alternative; $\frac{\overline{x} - x_{(1)}}{s} < c$, for testing a null hypothesis of normality against an exponential alternative; and

[8] Improved, smoothed Monte Carlo tables were later provided, both for testing normality and for testing exponentiality; see the discussion and references in Durbin (1975).

$\frac{\bar{x} - x_{(1)}}{x_{(n)} - x_{(1)}} < c$, for testing a null hypothesis of uniformity against an exponential alternative, where c is a suitably chosen critical value (Uthoff, 1970. pp. 1597-1598).

Shorack (1972) indicated proofs for the best scale invariant tests of exponentiality against one-sided Gamma and Weibull alternatives. In order to differentiate between the one-parameter exponential distribution and the Gamma distribution respectively, Shorack gave the uniformly most powerful scale invariant test of $H_0 : a = 1$ versus $H_1 : a > 1$, assuming a sample size n from a distribution with the density function,

$$f(x) = \frac{x^{a-1} \cdot \exp\left(-\frac{x}{\theta}\right)}{\theta^a \cdot \Gamma(a)}, \;\; x \geq 0, \;\; a > 0, \;\; \theta > 0 . \tag{2.14}$$

For this test, the null of exponentiality is rejected at significance level α if $T_n = \prod_{i=1}^{n} \left(\frac{x_i}{\bar{x}} \right) \geq k$, for k such that $P(T_n \geq k \mid a = 1) = \alpha$. A left-tailed test is also available for $H_1 : a < 1$. A caveat applies, as the test statistic, T_n is sensitive to errors near zero. For a Weibull distribution alternative, Shorack provided, for a sample with density

$$f(x) = \frac{a x^{a-1} \cdot \exp\left[-\left(\frac{x}{\theta} \right)^a \right]}{\theta^a}, \;\; x \geq 0, \;\; a > 0, \;\; \theta > 0 , \tag{2.15}$$

the most powerful scale invariant test of $H_0 : a = 1$ versus $H_1 : a > 1$, in which the null of exponentiality is rejected at significance level α if $T_a = \prod_{i=1}^{n} \left(\frac{\overline{x} \cdot x_i^{a-1}}{\overline{x^a}} \right) \geq k$, for k such that $P(T_a \geq k \mid a = 1) = \alpha$ (Shorack, 1972, pp. 213-214).

Puri and Rubin (1970) derived a characterization of the one-parameter exponential distribution, summarized in their Theorem 4: “Let X_1 and X_2 be two independent copies of a random variable, X with pdf $f(x)$. Then X and $|X_1 - X_2|$ have the same distribution, if and only if for some $\theta > 0$, $f(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right)$ for $x \geq 0$ and $f(x) = 0$ elsewhere” (Puri & Rubin, 1970, p. 2113).

Desu (1971) characterized the one-parameter exponential distribution, by presenting a function of order statistics which has the same distribution as the sample. If X is a random variable with nondegenerate distribution function (see Rohatgi, 2003), F, $X_1, ..., X_n$ is a random sample from that distribution, and $W = \min(X_1, ..., X_n)$, “then for each positive integer n, nW and X are identically distributed if and only if $F(x) = 1 - \exp(-\lambda x)$, for $x \geq 0$, where λ is a positive constant” (Desu, 1971, p. 837). See, also, Bury, who classified the exponentiality of the sample minimum, W, i.e., the first order statistic, as a reproductive property of the distribution (Bury, 1999, p. 181).

Characterization of the exponential distribution by properties of order statistics continued with Ahsanullah (1977), who gave a proof of the following Theorem:

> Let X be a nonnegative random variable with distribution function, F, that is absolutely continuous with respect to Lebesgue measure and strictly increasing on $[0,\infty)$. Then the following two statements are equivalent:
> (a) X has the one-parameter exponential density, Equation 2.1 above.
> (b) for sample size n, and i such that $1 \le i < n$, the statistics X and $(n-i)\left(X_{(i+1)} - X_{(i)}\right)$ are identically distributed, and, in addition, the cdf, F is either "new better than used" (NBU) or "new worse than used" (NWU), where these alternatives regarding $\bar{F} = 1 - F$ provide, respectively, that $\bar{F}(x+y) \le \bar{F}(x) \cdot \bar{F}(y)$ or $\bar{F}(x+y) \ge \bar{F}(x) \cdot \bar{F}(y)$ (Ahsanullah, 1977, pp. 580-581, paraphrased here).

One trend in the literature has been the analytical exploration of estimation of exponential distributions, under the rigor of Bayesian methods. Thus, Arnold (1970a) considered tests of hypothesis incorporating a preliminary test respecting one of two parameters involved. As will be seen, this is relevant to testing for exponentiality, wherein the mean and standard deviation are the same in the one-parameter case but different in the two-parameter, or right-shifted exponential distribution. In this context, the unknown shift - the location parameter in the two-parameter model – is treated as a nuisance parameter.

A criterion in the Bayesian approach is admissibility. Press explained that an estimator is admissible if there is no other estimator of the given parameter "for which the risk with respect to a loss function … is lower." When a statistician "must be concerned with the performance of estimators for many possible situational repetitions and for many values of the observables … admissibility is a reasonable Bayesian performance criterion.” However,

> in most other situations … statisticians are less concerned with performance of an estimator over many possible samples that have yet to be observed than they are with the performance of their estimator conditional upon having observed this particular data set and conditional upon all prior information available. For this reason, in non-experimental-design situations, admissibility is generally not a compelling criterion for influencing our choice of estimator (Press, 1989, pp. 27-29; see also Lehmann, 1959).

Arnold (1970b, 1973) demonstrated the inadmissibility of the best invariant estimator for the scale parameter, θ in Equation 2.4 above, of an exponential distribution, which he also refers to as “the usual scale estimate”, $T_1 = \frac{\sum_{i=2}^{n}\left(x_{(i)} - x_{(1)}\right)}{n}$, itself an improvement over the maximum likelihood and also minimum variance unbiased estimator, $T_0 = \frac{\sum_{i=2}^{n}\left(x_{(i)} - x_{(1)}\right)}{n-1}$. The usual estimate is shown to be inadmissible by reason of being dominated by an estimate which may correct for positive shift (and is no longer invariant), to wit,

$$T_2 = \begin{cases} T_1 & \text{if } X_{(1)} < 0 \\ \min\left(T_1,\ \frac{\sum_{i=1}^{n} x_{(i)}}{n+1} \right) & \text{if } X_{(1)} \geq 0 \end{cases}$$

The correction for positive shift may be explained by the fact that the second expression over which the minimum in T_2 is taken would be expected to overestimate the scale parameter in the case of such a shift (Arnold, 1970a, pp. 1261, 1263-1264).

Shapiro and Wilk (1972) extended their W-statistic, developed during the previous decade for testing normality, offering a scale-invariant test for exponentiality when both parameters must be estimated (with additional adjustments for the case of specified parameters). The authors defined the W-statistic as "the ratio of the square of an appropriate linear combination of the sample order statistics to the usual symmetric sum of squares about the mean" (1972, p. 355).

Deriving the W statistic for unspecified exponential origin (i.e., location) and scale parameters, the authors defined the 'standard' exponential distribution as one with values zero and unity, respectively, for these parameters, to wit, the simple exponential distribution with θ = 1. They proceeded to define, for a sample size n from the standard exponential distribution,

m_i = the expected value of the ith order statistic, $x_{(i)}$ (see Cox, 1964);

v_{ij} = the covariance of $x_{(i)}$ and $x_{(j)}$;

m' = the row vector of m_i's; and

V = the n by n, variance-covariance matrix of the v_{ij}'s.

Letting $y_i = \alpha + \beta x_i$, where the y_i's comprise a random sample from an exponential distribution with origin α and scale parameter β, and the x_i's comprise a corresponding random sample from the standard exponential, the authors present the "best unbiased estimate" of β, from generalized least squares (citation omitted), which is linear in terms of the order statistics, given in the form of vector and matrix products as: $\hat{\beta} = \dfrac{1'V^{-1}\left(1m' - m1'\right)V^{-1}y}{1'V^{-1}1m'\,V^{-1}m - \left(1'V^{-1}m\right)^2}$,

where $1'$ represents a row vector with n elements equal to unity. The explicit formulas,

$$m_i = \sum_{k=1}^{i}\left(n-k+1\right)^{-1}, \quad i = 1, 2, \ldots, n,$$
$$v_{ij} = \sum_{k=1}^{i}\left(n-k+1\right)^{-2}, \quad i \le j,$$
$$v_{ij} = \sum_{k=1}^{j}\left(n-k+1\right)^{-2}, \quad i > j,$$

are provided, followed by algebraic relationships among these moments, given as lemmas and corollaries, concluding with the simplified estimate,

$\hat{\beta} = \dfrac{n\left(\bar{y} - y_{(1)}\right)}{n-1}$. Describing the W-statistic as the normalized ratio of the square of $\hat{\beta}$ to $S^2 = \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$, the authors presented the statistic for a composite hypothesis of exponentiality (i.e., with unspecified parameters), as

$W = \dfrac{n\left(\bar{y} - y_{(1)}\right)^2}{\left(n-1\right)S^2}$, to be used as a two-tailed statistic, unless specific alternatives permit improving sensitivity by using a left- or right-tailed test (Shapiro & Wilk, 1972, pp. 356-357). Critical values were tabulated for W-exponential, using the Monte Carlo method with 5,000 samples for sizes n $\le$ 50 and $\frac{250{,}000}{n}$ samples for sizes, n from 51 to 100, using random generators in FORTRAN II, presented in 1965 by Fowlkes (citation omitted). Distributional results are presented, leading to adjustments for hypotheses with one or both parameters specified (1972, p. 358).

Although the treatment of loss functions in measure theoretic context[9] is beyond the scope here, Zidek (1973) considered an interesting exponential estimator, based upon a preliminary test for a zero location parameter, in other words, for whether an exponential distribution is one- or two-parameter in form. In this framework (developed independently of Arnold, 1970b; see Zidek, 1973, p. 266), the estimator for the scale variable of the exponential distribution is taken to be either $\bar{X}$ or $\bar{X} - \min(X_i) = \bar{X} - X_{(1)}$ (which is algebraically equivalent to T_1 in Arnold, 1970b), depending upon whether a preliminary test for the location parameter respectively indicates a value of zero for that parameter or does not.

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[9] The concepts of loss and risk functions, elements of assessing the quality of estimators, incorporated in the Bayesian approach as it became differentiated, and in information theory as well, were present in 19th Century precursors to, as well as in the early 20th Century development of inferential statistics. See references to Laplace and Gauss, with historical citations, in Lehmann (1959, pp. 26-27) and see the seminal article by Abraham Wald (1939). In an article representing the intersection of measure theory, statistical theory, the roots of Bayesian statistics, and information theory (formerly “communications theory”), citing such pioneers as Paul Halmos, R. A. Fisher and A. Kolmogorov, H. Jeffreys and L. J. Savage, and Claude Shannon and Norbert Wiener, Kullback and Leibler (1951) advanced the Kullback-Leibler information function (distinguished from the Fisher information function), measuring the mean information per observation for discriminating between hypotheses (see, also, Kullback, 1968; Zacks, 1971).

In the test for location, a value of zero is ‘accepted’ if and only if

$0 \le \frac{X_{(1)}}{\bar{X} - X_{(1)}} \le \frac{1}{n}$ (Zidek, 1973, pp. 264, 266). Brewster (1974), following the work of Arnold and that of Zidek, with whom he collaborated, continues a Bayesian analysis of loss and risk functions in order to provide an improved estimator in the nature of a step function.

The location parameter, or “unknown lower terminal”, was implicated as a nuisance parameter in a study by Durbin (1975), employing Fourier transforms along with K-S tests of goodness of fit. Further development was provided by Margolin and Maurer (1976). This particular direction of computationally intense estimation and testing lies outside the focus of the current study.

Stephens (1970) proposed modifications of eleven statistics for goodness of fit, so as to permit the use of abbreviated tables, based upon observed proportional relationships among critical values. The article includes a practical treatment of the calculation of several goodness of fit statistics in the form of K-S or C-vM statistics, used with data transformed to the uniform distribution, according to Seshadri, Csorgo, and Stephens (1969), discussed above, and of a class denominated periodogram statistics. Stephens’ modifications are presented in a table, listing each modified statistic along with a factor to achieve the desired simplification in tabulated significance values. For example, the factor for the K-S statistic, D, is applied to obtain the modified statistic, $D \cdot \left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}} \right)$. Other modification factors are similar in form but different in numerical detail. The respective modifications are presented along with upper and lower tail percentage points. This line of research continued, including work co-authored with Pettitt, utilizing Monte Carlo methods as well as exploring the asymptotic theory analytically, including the numerical solution of integral equations.

Pettitt and Stephens (1976) concentrated on statistics, including C-vM, A-D, and the Watson statistic, modified for use with censored data, when the hypothesized distribution is fully specified. Subsequent articles extended this theme to the case where the parameters are unknown or unspecified and must be estimated from censored data, and testing is for normality (Pettitt, 1976) or exponentiality (Pettitt, 1977).

Pettitt (1977) gave asymptotic distributions of two C-vM type statistics for testing for exponentiality when the scale parameter must be estimated, and the data is censored. The second statistic is differentiated from the first by inclusion of the A-D weight function (see above). The two statistics are identified by the customary notation, respectively, ${}_{p}W^{2}=\int_{0}^{p} z^{2}(t)dt$ and ${}_{p}A^{2}=\int_{0}^{p} z^{2}(t)\left(t-t^{2}\right)^{-1}dt$, with the advance-subscript specifying the uncensored proportion of data which has been transformed to the uniform distribution on [0, 1] (see Pettitt & Stephens, 1976), and where z(t) is a Gaussian process "with mean zero and covariance function given for $0 \le s,t \le p \le 1$ by

$\text{cov}\{z(t), z(s)\} = k(s,t) = \min(s,t) - st - p^{-1}\{(1-t)\ln(1-t)\}\{(1-s)\ln(1-s)\}$" (Pettitt, 1977, p. 629).

In the case of a fixed number, r of observations (failure- or type II censoring), in testing the null hypothesis that observations come from the one-parameter exponential distribution, the specific statistics which may be used are ${}_r\hat{W}_n^2 = n\int_0^{\hat{t}_r}\{\hat{F}_n(t)-t\}^2 dt$ and ${}_r\hat{A}_n^2 = n\int_0^{\hat{t}_r}\{\hat{F}_n(t)-t\}^2 (t-t^2)^{-1} dt$, respectively, where $\hat{t}_i = F(x_i, \hat{\theta})$ and $\hat{F}_n(t)$ is the empirical distribution function of the $\hat{t}_i$'s for $t \le \hat{t}_r$. The scale parameter in this case is estimated (as in Jackson,1967, above) by $\hat{\theta} = \dfrac{\sum_{i=1}^{r} x_i + (n-r)x_r}{r}$.

When a random number of observations are censored, with R known observations $0 \le x_1 \le ... \le x_R \le x_p$ (time- or type I censoring), then the statistics take the form, ${}_p\hat{W}_n^2 = n\int_0^{\hat{p}}\{\hat{F}_n(t)-t\}^2 dt$ and ${}_p\hat{A}_n^2 = n\int_0^{\hat{p}}\{\hat{F}_n(t)-t\}^2 (t-t^2)^{-1} dt$, respectively, with $\hat{p} = F(x_p; \hat{\theta})$, and θ is estimated by $\hat{\theta} = \dfrac{\sum_{i=1}^{R} x_i + (n-R)x_p}{R}$ (Pettitt, 1977, p. 629).

The author outlined a proof that the proposed statistics converge in distribution to ${}_pW^2$ and ${}_pA^2$, respectively. Then, treating the asymptotic theory, the author indicated approximations based upon the calculation of eigenvalues of the integral equation, $\int_0^p k(s,t)\, f(s)\, ds = \lambda f(t)$ for $0 \le t \le p$. Citing earlier asymptotic theory and referring to Pettitt (1976) and Pettitt and Stephens (1976 for computing formulae, the author presented percentage points of the proposed statistics for several values of p, claiming these to provide reasonable approximations for n "larger than about 20" and r = .5n or greater, with r = pn (Pettitt, 1977, pp. 631-632).

Hollander and Proschan (1972) considered testing exponentiality against the "new better than used" (NBU) alternative class of life distributions. If $\bar{F}(x) = 1 - F(x)$ represents the survival cdf, then a life distribution is NBU (or positive aging) if $\bar{F}(x+y) \le \bar{F}(x) \cdot \bar{F}(y)$ for all positive x and y, with equality characterizing the exponential distribution, under which used items are no better than, nor worse than new items. For the NBU class, a new unit has greater probability of surviving to age x, $\bar{F}(x)$, than does a unit which has survived to age y, for which the probability of surviving an additional time, x is $\dfrac{\bar{F}(x+y)}{\bar{F}(y)}$. Thus, Hollander and Proschan test $H_0 : F(x) = 1 - \exp\left(-\dfrac{x}{\theta}\right)$ against $H_1 : F$ is NBU (and not exponential), which is analogous to testing against the alternative of an IFR distribution, a smaller class of alternatives contained in the NBU class, which includes fluctuating failure rate distributions as well. For convenience, the authors utilize a scale-invariant U-statistic which is asymptotically equivalent to a direct test of the NBU criterion (1972, pp. 1136-1137). The null distribution is derived, power is discussed relative to IFR and other NBU alternatives, and critical values are tabulated by the Monte Carlo method, with 100,000 replications for sample sizes under 20 and 10,000 replications for 20 and above (1972, p. 1145).

Basu and el-Mawaziny (1978) investigated the reliability function for k-out-of-m systems, which operate if and only if at least k of the m system components are operating successfully. The reliability function gives the probability that a device will survive an operating period of at least x and equals unity minus the time-to-failure distribution function (Bury, 1999, p. 193). A special case treated, *inter alia*, by the authors is that of components with identically, independently, exponentially distributed failure times, all with the same parameter, θ. In this case, the reliability of the system, $R(t)=\Pr(X>t)$ = Pr(k or more components fail at time greater than t) simplifies to $R(t)=\sum_{j=k}^{m}\sum_{\alpha=0}^{m-j}(-1)^{\alpha}\binom{m}{j}\binom{m-j}{\alpha}\left[1-F(t)\right]^{j+\alpha}$ $=\sum_{j=k}^{m}\sum_{\alpha=0}^{m-j}(-1)^{\alpha}\binom{m}{j}\binom{m-j}{\alpha}\exp\left\{\frac{-t(j+\alpha)}{\theta}\right\}$. The authors then compared the minimum variance unbiased estimator (MVUE) with the maximum likelihood estimator (MLE) of the reliability function at time, t. The MVUE is given by $\hat{R}(t)=\sum_{j=k}^{m}\sum_{\alpha=0}^{m-j}(-1)^{\alpha}\binom{m}{j}\binom{m-j}{\alpha}\left\{\max\left[1-\frac{(j+\alpha)t}{T}\right],0\right\}^{r-1}$, where T is the total lifetime. The MLE is given by $\tilde{R}(t)=\sum_{j=k}^{m}\sum_{\alpha=0}^{m-j}(-1)^{\alpha}\binom{m}{j}\binom{m-j}{\alpha}\exp\left\{\frac{-t(j+\alpha)}{\bar{Y}}\right\}$, where the MLE of θ is given by $\tilde{\theta} = \overline{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$ for a complete random sample of failure times, $Y_i, \ldots Y_n$ (Basu & el-Mawaziny, 1978, pp. 851-852).

Relative performance was investigated in terms of mean squared error (MSE) and bias, presented in tables for sample sizes n = 3, 4, 5, and 10, for three values each of t and θ, and for different combinations of m and k, using Monte Carlo simulation. The authors concluded that “the MVUE and MLE of R(t) are asymptotically equivalent.” Whereas the MVUE is unbiased by theory, the estimated bias was used as a check on computations. The authors found both bias and MSE to be “quite small and comparable” and conclude: “thus it seems, even in small samples, that either the MLE or the MVUE of R(t) can be used” (Basu & el-Mawaziny, 1978, pp. 853).

In a series of articles (Gail & Gastwirth, 1978a; 1978b; and Gail & Ware, 1978), Gail and collaborators treated a scale-free goodness-of-fit test based upon the Lorenz curve and associated Gini index, used as a measure of income inequality in economics.

The Lorenz curve has similarities to a relative cumulative frequency ogive (see, for example, Daniel, 1999, pp. 87-89). The ogive is a continuous, broken-line graph from grouped data, with the observed random variable on the horizontal axis and with range [0, 1] on the vertical axis, representing cumulative relative frequency. An ogive is, thus, a linearly-interpolated representation of the empirical distribution function. Any percentile of the empirical distribution may be approximated by finding the corresponding point in [0, 1] and projecting a segment horizontally from that point on the vertical axis to the ogive, and then vertically to the horizontal scale of the observed variable. In the case of a Lorenz curve, the horizontal axis, rather than showing the measurement scale of the observed variable, reflects the relative cumulative frequency of the data, with frequency accumulating over order statistics or over ascending class intervals, in the typical case in which data is grouped. Thus, both the vertical and horizontal scales cover unit intervals. The vertical scale represents the cumulative proportion of the sample total represented by the sum of order statistics or of grouped data corresponding to any percentile, or any quantile in general of the empirical distribution. In the economic context,

> the [Lorenz] curve is constructed by plotting the cumulative percentage of families at or below a given income level and the cumulative percentage of total personal income received by these families. … Equality of income would result if each family received an equal proportion of the total income, so that the bottom 20% would receive 20% of the total income, the bottom 40% would receive 40%, and so on. The Lorenz curve representing this would have the equation y = x. (Harshbarger & Reynolds, 2004, p. 945).

Typically, the Lorenz curve will reflect income inequality and appear as a concave-up curve between the origin and the upper right-hand corner of the unit square. The area above the Lorenz curve and below the line connecting those two points, in other words, the identity function, y = x, is utilized in an index of income inequality, to wit, the Gini index, which equals that area divided by the area below the line, y = x, the latter area being 0.5. Thus, the Gini index is given by $\frac{\text{area between y = x and y = L(x)}}{\text{area below y = x}} = 2\left(\text{area between y = x and y = L(x)}\right)$, where L(x) represents the Lorenz curve (Harshbarger & Reynolds, 2004, p. 945).

For a goodness-of-fit test for the one-parameter exponential distribution, Gail and Gastwirth (1978b) defined the Lorenz statistic by the ordinate of the Lorenz curve for the lower proportion, p of the sample values:

$$L_n\left(p\right) = \frac{\sum_{i=1}^{r=[np]} X_{(i)}}{\sum_{i=1}^{n} X_{(i)}}, \text{ for } 0 < p < 1. \tag{2.16}$$

For the case of the two-parameter exponential distribution, the authors provide a modified statistic, $L_n^*\left(p\right) = \frac{\sum_{i=1}^{r=[np]}\left(X_{(i)} - X_{(1)}\right)}{\sum_{i=1}^{n}\left(X_{(i)} - X_{(1)}\right)}$, employing the first order statistic to estimate restoration of a location parameter of zero. In the case the null distribution is the one-parameter exponential, the population Lorenz curve is defined by $\lambda\left(p\right) = \frac{1}{\theta}\int_0^p\left[-\theta\ln\left(1-t\right)\right]dt = p+\left(1-p\right)\ln\left(1-p\right)$ (1978b, p. 787).

The authors furnished a table with exact percentiles of the statistic, $L_n\left(.5\right)$ for n = 2 through 40, after deriving the exact distribution of the statistic. They demonstrated convergence of their statistic to $\lambda\left(p\right)$ and consistency of the test and proceeded to investigate asymptotic relative efficiency (ARE) and power of the test with various alternatives.

The authors' Monte Carlo method entailed comparing the error rate and power of the Lorenz statistic with ten other goodness of fit tests for exponentiality, against seven alternative distributions including the Weibull, Uniform, Pareto, shifted Pareto, shifted exponential (i.e., two-parameter), and gamma, with 1000 simulated random samples of size 20. A random generator in IBM 360 assembly language was used, along with single-precision programming in FORTRAN IV. The authors stated that the Type I error rate table "shows that each test rejected approximately five percent of the exponential samples as theory predicts," but this is less true of the Weibull alternative with parameter, 0.8, and the Pareto, and the shifted exponential parameters as well, as one may tell from the published table. The authors classified scale-free goodness-of-fit tests for exponentiality into five categories: (1) tests based on the empirical distribution function; (2) tests based on unordered uniform spacings; (3) tests based on normalized exponential spacings; (4) tests on uniform order statistics; and (5) other promising tests (including the Lorenz statistic, which the authors describe as "the sum of ordered uniform spacings". Their conclusions have empirical as well as heuristic value, for quoting at length:

> The Lorenz statistic usually has greater power than the Durbin KS test (category 1), the IFR[10] test (category 3), and the Shapiro-Wilk test (category 3) … . The Lorenz statistic outperforms C^+, C^-, and $|C|$ (category 4) against the important Weibull and gamma alternatives. The Moran

[10] Related acronyms are IFRA for "increasing failure rate average", and DFR and DFRA, for the 'decreasing' cases (see, for example, Zacks, 1971, pp. 524-528).

> statistic (category 2) exhibits slightly greater power than $L_n(.5)$ against these two alternatives. However [citation omitted] … the Moran statistic is so sensitive to recording errors in small observations as to limit its practical usefulness, whereas $L_n(.5)$ is insensitive to such errors [citation omitted]. In summary, the Lorenz curve goodness-of-fit test has desirable power properties against a variety of alternatives, is readily calculated, and is insensitive to minor errors in the small observations (Gail & Gastwirth, 1978b, pp. 790-792).

Gail and Gastwirth (1978a) proceeded from the Lorenz statistic to the more powerful Gini statistic, which depends on the entire Lorenz curve, rather than a single point (p = 0.5). The Gini index, the basis of the statistic, is defined as “twice the area between the equiangular line and the population Lorenz curve” (1978b, p. 350). The Gini statistic, for exponentially distributed data is given as

$$G_n = \frac{\sum_{i=1}^{n-1}\left\{i(n-i)\left(X_{(i+1)} - X_{(i)}\right)\right\}}{(n-1)\sum_{i=1}^{n} X_i}$$

, where “unit exponentiality” is assumed without loss of generality, because G_n is scale-free. The exact distribution of G_n for a sample from the unit exponential population is

$$P(G_n \le x) = x^{n-1}\left\{\prod_{i=1}^{n-1} c_i\right\} - \sum_{j=r+1}^{n-1}\left(x - c_j\right)^{n-1}\left\{c_j \prod_{k\ne j}^{n-1}\left(c_k - c_j\right)\right\}^{-1}$$

, where $c_j = \dfrac{n-j}{n-1}$ and r is the largest index such that $x \le c_r$. A more convenient form is given for x large and r small. A table is provided with percentiles (95, 97.5 and 99) of the Gini statistic for n from 3 to 20 (1978a, p.351-2). The Gini statistic has a historical precursor in “Gini’s mean difference” which differs from the former by a factor of $2\bar{X}$ and is related to Downton’s unbiased estimator of the standard deviation for normal populations, given by David as $"\sigma" = \dfrac{2\sqrt{\pi}}{n(n-1)}\sum_{i=1}^{n}\left[i-\tfrac{1}{2}(n+1)\right]X_{(i)}$ (David, 1970, pp. 146-147, citations therein omitted here.)

For comparison of power using Monte Carlo simulation, with 1,000 samples size n = 20, the authors considered seven alternative distributions and compare the Gini statistic to the Lorenz statistic, the Pietra statistic, $P_n = \dfrac{\sum_{i=1}^{n-1}\left|X_i - \bar{X}\right|}{2n\bar{X}}$, and a scale-free statistic similar to the scale- and location-free Shapiro-Wilk test, $R_n = \bar{X}\left\{\dfrac{n-1}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}\right\}^{\frac{1}{2}}$. The alternatives are Weibull with shape parameters 0.8 and 1.5, the Uniform distribution on (0, 2), Pareto and shifted Pareto distributions, shifted exponential, and gamma. The Gini statistic outperforms the others except in three alternative and competitor combinations, in which the power for Gini is comparable (Gail & Gastwirth, 1978a, p. 351).

Gail and Ware (1978) focused on a concern for sensitivity of tests for exponentiality to data contaminated by small measurement errors, with the objective of identifying tests which are insensitive to measurement error, in particular, to rounding and truncation error respectively. This article was written close to the time of Gail and Gastwirth (1978a, 1978b), in which the related Gini and Lorenz statistics were treated, respectively. Monte Carlo estimates of sensitivity to measurement error are given for goodness-of-fit tests including

Lorenz and Gini statistics, forms of the K-S test. Shapiro-Wilk, and the least satisfactory of the group, Moran's statistic, with some modification. The Monte Carlo study utilized 1,000 samples of size, n = 20. It was found that truncation distorted test results more than rounding error for most of the tests. The Lorenz statistic and Gini statistic were "remarkably insensitive to rounding error" (Gail and Ware, 1978, p, 307). Other conditionally good performers were Durbin's form of the K-S statistic, IFR, Shapiro-Wilk, and a trimmed Moran statistic. The authors also discuss power against gamma and Weibull alternatives.

Thall (1979) adapted Huber's location parameter theory to estimation of the parameter of an exponential distribution. Langenberg and Srinivasan (1979) focused on distinguishing the exponential distribution from decreasing mean residual life alternatives. They derived the exact null distribution for a test statistic proposed by Hollander and Proschan and tabulated critical values for n = 2 to 60. Hines and Hines (1979) provided a table of critical values for testing the null hypothesis that Euclidean distances, X_i, from m randomly selected sampling locations, O_i, are exponentially distributed, using Eberhardt's statistic, which is given as $A = \frac{m\sum_{j=1}^{m} X_j^2}{\left(\sum_{j=1}^{m} X_j\right)^2}$ (Langenberg & Srinivasan,1979, p. 73).

The 1980s.

Lin and Mudholkar (1980) presented a bivariate F-test for exponentiality which is effective against alternative distributions with both monotonic and non-monotonic hazard rate functions. The authors observed that Gnedenko's one-tailed F-test is "well-suited to detect alternatives such as Weibull … [citation omitted], or gamma which have nonconstant but monotone hazard rates", but less suitable for other alternatives, whereas a two-tailed version of the F-test due to Harris (citation omitted) was proposed as being "superior against the lognormal distribution which has a U-shaped hazard rate function … [but] is inferior in detecting a monotone hazard rate" (1980, p. 80). Gnedenko's F-test, utilizing a reproductive property of the gamma distribution under which a sum of exponential variables has a Chi-square distribution, compares the first r with the last n-r normalized spacings from a sample subjected to a null hypothesis of exponentiality. The two-tailed version compares sequences of order statistics combined from the beginning and end of the data with the sequence in the middle. Lin and Mudholkar's statistic develops these ratios in two separate F-variables, respectively comparing the upper and lower sums, with the middle sum, and joins the variables in a bivariate F-distribution (Lin & Mudholkar, 1980, p. 80). The authors concluded from a Monte Carlo study that their test "offers a good protection against the nonmonotone failure rate alternative, viz. the lognormal distribution, without a significant loss of power against the monotone hazard rate distributions". The Monte Carlo experiment was implemented on a

UNIVAC 1100/40 computer, with 1,000 samples used for each alternative, with sample sizes, n = 20 and n = 30. The alternative distributions were Chi-square with 1, 3, 4, and 8 degrees of freedom, lognormal with parameters 0.6, 0.8, 1.0 and 1.2, Weibull with parameters 0.5 and 2.0, and Beta (1, 2), respectively (1980, p. 81).

The distributional background for Lin and Mudholkar's test began with the property that the normalized spacings, $D_{ni} = (n-i+1)\left(X_{(i)} - X_{(i-1)}\right)$ with $X_{(0)} \equiv 0$ derived from an exponential distribution have the same exponential distribution as the original X_i's (Lin & Mudhokar, 1980, p. 79; and see Gnedenko, Belyayev, and Solovyev, 1969, p. 236; Pyke, 1965, p. 400). Both Chi-square and exponential distributions are special cases of the gamma distribution, to wit, $g(x;\alpha,\beta)$ with shape parameter, α, and scale parameter, β (see, for example, Bury, 1999; Miller & Miller, 1999). With shape parameter of unity, the gamma density becomes one-parameter exponential with mean, $\theta = \beta$, the scale parameter. With scale parameter equal to two, the gamma density becomes Chi-square, with degrees of freedom, $\nu = 2\alpha$.

Moreover, the gamma distribution has the "reproductive property" that the product of a constant, c, and the sum of k gamma-distributed random variables with respective shape parameters, α_i, and all with the same scale parameter, β, itself has a gamma distribution with shape parameter equal to $\sum_{i=1}^{k} \alpha_i$ and scale parameter equal to $c\beta$ (Bury, 1999, p. 211). In order to obtain a Chi-square variable through such a construction, we choose $c = \frac{2}{\theta}$, where θ represents the distribution parameter under a null hypothesis of exponentiality. If a sum, S, is constructed of k iid exponential variables, each equivalent to a gamma variable with density, $g\left(x; 1, \theta\right)$, then $\frac{2}{\theta} \cdot S$ has the gamma distribution with scale parameter $\frac{2}{\theta} \cdot \theta = 2$ and shape parameter $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} 1 = k$. Because the scale parameter equals two, this distribution will be the same as Chi-square with k degrees of freedom. Thus, a ratio of two such variables, each divided by its respective degrees of freedom will constitute a random variable from the corresponding F-distribution.

Lin and Mudholkar, improving upon Gnedenko et al. and Harris, separately compared the sum of the lower r normalized spacings with the sum of the middle n-2r spacings and the sum of the upper r spacings with the middle sum, these sums being $S_{l[ower],r} = \sum_{i=1}^{r} D_{ni}$, $S_{m[iddle],r} = \sum_{i=r+1}^{n-r} D_{ni}$, and $S_{u[pper],r} = \sum_{i=n-r+1}^{n} D_{ni}$. F-statistics follow, for the lower and upper comparisons, to wit: $F_l = \frac{S_{l,r} / r}{S_{m,r} / \left(n-2r\right)}$ and $F_u = \frac{S_{u,r} / r}{S_{m,r} / \left(n-2r\right)}$. The authors proposed rejecting a null hypothesis of exponentiality if either F_l or F_u falls outside a real interval, (a, b), under the bivariate F-distribution, these critical values being determined with the aid of a theorem due to Hewett and Bulgren (1971), which provides the inequality,

$P\left\{ a \le F_l \le b,\ a \le F_u \le b \mid H_0 \right\} \ \le \ \left[P\left\{ a \le F \le b \right\} \right]^2$, where the F-distribution on the right side has numerator and denominator degrees of freedom of, respectively, 2r and 2(n - 2r) degrees of freedom (Lin & Mudholkar, 1980, p. 80). The authors' simulation studies, as well as those of Hewlett and Bulgren, "show that the inequality [above] is very sharp. The authors proceeded to describe the procedure:

> Johnson and Kotz (1972, p. 241) observe that the 'right-hand side of [the inequality above] is quite a good approximation to the left' and that the accuracy increases as the denominator degrees of freedom increases and the numerator degrees of freedom decreases. The cutoff points a and b may now be determined by setting the right-hand side [of the inequality above] equal to $1-\alpha$, where α is the level of significance, and assuming equal tail probabilities of the F distribution. It is easy to see that the bivariate F tests thus constructed is conservative in the sense that the probability of type I error is smaller than the nominal level. The difference, however, is negligible because of the quality of the inequality (Lin & Mudholkar, 1980, pp. 80-81).

Kimber and Stevens (1981) investigated the maximum likelihood ratio test for discordancy of upper outliers in a sample from the one-parameter exponential distribution, with order statistics $y_1, ..., y_{n-k}$ taken as from the distribution with parameter θ and $y_{n-k+1}, ..., y_n$ taken as from the distribution with parameter $\theta\lambda$ and hypotheses, $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$.[11] The authors limited their treatment

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[11]The parameter, discordancy factor, and alternative hypothesis have been transformed to be consistent with the algebraic expression of the exponential cdf adopted herein.

to k = 2 large outliers. Evidence against the null hypothesis is provided by a large value of the test statistic, $T_k = \dfrac{y_{n-k+1} + y_{n-k+2} + ... + y_{n,}}{\sum_{i=1}^{n} y_i}$, of which, for k = 2, the null distribution is derived and tabulated, for values of n between 5 and 20 and $\alpha = .01$ and $.05$. Both the pdf and cdf are given in closed form (1981, p. 153).

Performance of T_k is compared to that of the W-test of Shapiro and Wilk, as adapted by Stephens “for use when only the scale parameter … is unknown” , with test statistic $W = \dfrac{\left(\sum x_i\right)^2}{n\left\{(n+1)\sum x_i^2 - \left(\sum x_i\right)^2\right\}}$ (citations omitted; Kimber & Stevens, 1981, p. 155). The authors discussed analysis of outliers using sensitivity contours, and find that the statistic “T_2 is superior to W when up to two large values are present in a sample” (1981, p. 156).

Citing work by Pettitt and Stephens on the C-vM statistic (Pettitt, 1976; Pettitt & Stephens, 1976) but, without explanation, not citing Pettitt (1977) which also treats this statistic (as well as Anderson-Darling; see above) for exponential data which is censored and for which the scale parameter must be estimated, Sirvanci and Levent (1982) retraced some of the same analytical background, with incidentally complementary detail. Type I censoring is considered, testing one-parameter exponentiality. As treated in the earlier articles, modification of the C-vM statistic is required, as, in its original form, that statistic “can only be used with complete samples, … and for simple H_0, where θ … is specified” (Sirvanci &

Levent, 1982, p. 642). Using the empirical cdf, data is first transformed to a distribution which is uniform, under the null hypothesis. The modifications treated in Pettitt and Stephens (1976) and Pettitt (1976) are utilized, with the empirical distribution, its parameter estimated by the maximum likelihood estimator,

$\hat{\theta} = \dfrac{\sum_{i=1}^{N} X_i + (m-N)T}{N}$, where m is the number of items tested, T is the predetermined recording duration, and N is the number of failures observed in the interval, [0, T]. The asymptotic distribution for the test statistic is derived and critical percentiles of the statistic are tabulated for values of p, the cdf value at time of censoring and α = .01, .025, .05, and .1.

Deshpande (1983) considered tests for exponentiality against a particular class of alternatives, to wit, distributions with increasing failure rate average. The test statistics proposed are U-statistics (see Lehmann, 1998; Zack, 1971), which are asymptotically normal. Increasing failure rate average distributions comprise “the smallest class of probability distributions which contains the exponential distribution and is closed under formation of coherent systems” and “arise as life distributions from various useful shock models” (Deshpande, 1983, p. 514). If $F(0)=0$ for the cdf, $F(x)$, and $\bar{F}(x)=1-F(x)$, then F is an increasing failure rate average distribution if and only if $\bar{F}(bx) \geq \{\bar{F}(x)\}^b$, for x > 0 and 0 < b < 1, with equality characterizing an exponential distribution (1983, p. 514).

The author developed a class of U-statistics, denominated J_b statistics, which utilize the foregoing inequality to measure the deviation of a sample distribution from the null hypothesis of exponentiality and employ the Wilcoxon statistic for the observed x_i's, compared with the $b \cdot x_i$'s, where b is a fixed number in the interval, (0, 1). The author employed a Monte Carlo study, using 10,000 samples for each size and condition treated, to estimate power with respect to two alternatives, the Weibull distribution with $F(x) = 1 - \exp(-x^2)$ and a linear failure rate distribution with $F(x) = 1 - \exp\left(-x - \frac{1}{2}x^2\right)$, both distributions having increasing failure rate average. Exact critical values are tabulated as well. Pitman asymptotic relative efficiency is calculated with respect to Hollander & Proschan's test and to three distribution families, including the Makeham distribution as well as the Weibull and linear failure rate distributions, each expressed in representations that reduce to the null distribution when the given parameter vanishes. The J_b tests are shown to have "fairly high efficiency" and have other superior qualities (Deshpande, 1983, pp. 515-518).

Angus (1982) investigated tests for exponentiality which are directed towards gauging deviation from the loss of memory property of the exponential distribution. Polynomial smoothing is used with Monte Carlo simulation and asymptotic theory, to tabulated 90th and 95th percentiles for the proposed statistics, for n = 10 to 50. The alternative hypotheses investigated for power include members of the Chi-square, lognormal, Weibull, Pareto, shifted Pareto, shifted exponential, and Beta distribution families. The competing tests studied were the Lorenz test (Gail & Gastwirth, 1978a), two versions of the K-S test, the increasing/decreasing hazard rate test (or “increasing failure rate” – IFR) of Proschan and Pyke, C-vM, the Moran test, and three F-tests of Gnedenko and others (citations omitted). The Monte Carlo technique employed 1,000 samples, sizes 20 and 30, for power and 10,000 samples for critical values (Angus, 1982, p. 247).

Another historical watershed is marked by an overview by Spurrier (1984), treating developments subsequent to Epstein’s papers of 1960 (see above). Four broad categories are used for grouping the tests treated: 1) tests based on estimators of the distribution function and on probability plots; 2) tests resulting from transformations of exponential data; 3) specialized tests for increasing or decreasing failure rate and related alternatives; and 4) tests for specific alternatives (Spurrier, 1984, p. 1637).

Epstein (1960a) suggested plotting the inverse exponential pdf (the cdf subtracted from unity) applied to the empirical percentile against the data, but a test of hypothesis for exponentiality required that such a distance be formalized in a distance statistic. The first such tests were proposed by Lilliefors (1969) based upon the K-S statistic and the same year by van Soest (citation omitted), using the C-vM statistic (Spurrier, 1984, pp. 1638-1639). Finkelstein and Schafer (1971), Spurrier noted, suggested an improved, K-S-type statistic for testing for exponentiality, with unknown mean, $S_n^* = \sum_{i=1}^{n} \delta_i^*$, where

$\delta_i^* = \max\left[\left| F^*\left\{X_{(i)}\right\} - \frac{i-1}{n} \right|, \left| F^*\left\{X_{(i)}\right\} - \frac{i}{n} \right| \right]$, and $F^*(X) = 1 - \exp\left(-\frac{X}{\bar{X}}\right)$. Critical values were obtained by Finkelstein and Shafer, using 25,000 Monte Carlo runs for each sample size. The authors stated that their statistic was more powerful than that proposed by Lilliefors, based upon K-S, and equivalent to, but simpler in computation than, Soest's C-vM-like statistic (Finkelstein & Schafer, 1971, pp. 642-644; Spurrier, 1984, p. 1639). In a 1974 article, Stephens modified several K-S and C-vM type statistics for exponentiality, freeing critical points from dependence upon sample size (citation omitted). Other work was done in the theoretical arena, including some mentioned herein, above. Srinivasan reported much higher power for a modification of a K-S type statistic than achieved for Lilliefors' statistic, but this was controverted by other researchers (citations omitted; Spurrier, 1984, pp. 1639-1640).

Spurrier gave Sarkadi's (1975) test for two-parameter exponentiality as an example of the approach of measuring the strength of the linear relationship between order statistics for the data and their expected values. Sarkadi observed similarity between the Shapiro-Wilk Test (Shapiro & Wilk, 1972) and the Shapiro-Fancia test (citation omitted) for normality, noting that the latter statistic is easier to compute, and presented a version of the latter for testing exponentiality (Sarkadi, 1975, p. 445; Spurrier, 1984, p. 1640). Sarkadi presents this test statistic (using lower critical values) as $W'' = \frac{\left\{ \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{X_{i:n}}{n-j+1} - \sum_{i=1}^{n} X_{in} \right\}^2}{\sum_{i=1}^{n} \left(X_{in} - \bar{X}_n \right)^2}$. Sarkadi remarks that the Shapiro-Wilk and Shapiro-Francia tests have the same asymptotic distribution apart from a scale factor (1972, p. 449). Spurrier, crediting Sarkadi, comments on the inconsistency of the Shapiro-Wilk test, in respect to a beta alternative with parameters $\sqrt{2}\ -1$ and unity, respectively (Spurrier, 1984, p. 1641).

In the second category, Spurrier discussed transformation of data prior to testing for exponentiality, which is done for several purposes. In some cases, transformation allows for analysis of censored data without special tables. One transformation permits testing for two-parameter exponentiality using methods developed for simple exponentiality. Other transformations lead to test statistics distributed as the order statistics of a uniform distribution on [0, 1]. Yet others achieve increased power for certain alternatives. Spurrier added that some of these transformations were developed independently by several authors, rendering it difficult to acknowledge an original source (Spurrier, 1984, p. 1642).

A common transformation is that to normalized spacings, $D_i = \left(n+1-i\right)\left[Y_{(i)} - Y_{(i-1)}\right]$, with $Y_{(0)}$ set equal to zero. For a population with the simple exponential distribution, these spacings follow the same distribution as the population, the exponential being the only nonnegative distribution with this property. Tests for complete samples can be applied in the case of type II censoring (i.e., after the rth failure, with r predetermined), because the spacings in the latter case are distributed, under a hypothesis of exponentiality, as a complete exponential sample, size r (Spurrier, 1984, pp. 1642-1643; citations to other authors omitted).

Related to normalized spacings is the total-time-on-test transformation, with $T_1 = nY_{(1)}$ and $T_i = \sum_{j=1}^{i-1} Y_j + (n-j+1)Y_{(i)}$, for i = 2, 3, … n. For a simple exponential distribution, the resulting distribution of T_1 is that of the order statistics from a uniform distribution on [0, T^*], where T^* is the preassigned total life, under type I censoring. In the case of type II censoring (and also for complete samples), the distribution is that of order statistics of a sample size i–1, from the uniform distribution on [0, T_i], where the test is terminated on the ith failure (Spurrier, 1984, p. 1643; citations to other authors omitted).

Spurrier noted a transformation proposed by Wang and Chang (1977),

$z_j = \left[\frac{\sum_{i=1}^{j} Y_{(i)}}{\sum_{i=1}^{j+1} Y_{(i)}} \right]^j$ for j = 1, … , n-1, which, under simple exponentiality, yields a uniform distribution on [0, 1], a characterization of the one-parameter exponential distribution. The authors proposed a test statistic based on the transformation,

$\chi^2 = -2\sum_{j=1}^{n-1} \ln\left[g\left(z_j\right)\right]$, where $g\left(z_j\right) = 2z_j$ for $0 \le z_j < \frac{1}{2}$ and $g\left(z_j\right) = 2\left(1 - z_j\right)$

otherwise. The statistic has a chi-square distribution with 2n – 2 degrees of freedom, for data from the one-parameter exponential distribution, and outperforms the Lilliefors test based on K-S, for Weibull and gamma alternatives (Spurrier, 1984, pp. 1643-1644).

Spurrier mentioned Durbin’s (1975) transformation for data under a two-parameter exponential hypothesis, yielding a sample sized n-1, omitting the minimum transformed to zero, to which tests for one-parameter exponentiality can be applied. The transformed data was obtained simply by subtracting the sample minimum (i.e., the first order statistic) from order statistics 2 through n (Spurrier, 1984, p. 1644).

Spurrier’s next category encompassed tests sensitive to broad classes of alternatives important in reliability testing[12]. These included, respectively, increasing and decreasing failure rate (IFR and DFR), increasing and decreasing failure rate average (IFRA and DFRA), and new better than used (NBU) classes of alternative distributions. Works by Barlow and Proschan and by Bickel and Doksum, respectively, on IFRA alternatives were mentioned. Work on the Lorenz curve and Gini statistic by Gail and Gastwirth was also mentioned, with caveats regarding the want of consistency for all alternatives (citations omitted; Spurrier, 1984, pp. 1644-1646).

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[12] For a comprehensive treatment of classes of life distributions, see Hollander and Proschan (1984).

Asymptotically most powerful linear rank tests were briefly discussed. Bickel and Doksum concluded that rank tests are less powerful than a specialized class of scale-invariant, ratio statistics (Spurrier, 1984, p. 1647). Spurrier discusses tests which group normalized spacings by order of occurrence, using statistics with an F-distribution, including work by Gnedenko, Belyayev, and Solovyev (1969), and work, respectively by Lin and Mudholkar and by Harris (citations omitted; Spurrier, 1984, p. 1647; and see the discussion of Lin & Mudholkar, 1980, above). Gnedenko et al. discussed an example of such a test and of the need to utilize a sufficiently large sample to draw meaningful inferences about the failure rate. The F-test permits a two-sided test, in which a left-tailed rejection indicates an increasing failure rate, while a right-tailed rejection indicates a decreasing failure rate (Gnedenko et al., 1969, pp. 236-238).

Specialized tests were treated summarily by Spurrier (1984, pp. 1647-1648). A paragraph was dedicated to tests for specific alternatives. Spurrier concluded, remarking on the breadth of this field of investigation:

> A large number of tests for exponentiality have been proposed in the literature. The user may feel overwhelmed when trying to decide on which test to use. It is clear that one should consider the types of alternatives that are of interest. Then one should select a test that is sensitive to those alternatives (Spurrier, 1984, p. 1649).

Noting that although the literature is wide, “there has not been a comprehensive study comparing all tests applicable to the important families of alternatives,” Spurrier issues a caveat respecting goodness-of-fit tests:

> In tests of fit, one often wishes to validate the null hypothesis. Thus, it is important to consider the probability of making a type II error for reasonable alternative models. Unfortunately, for most tests of exponentiality and most alternatives there are no practical ways to make these calculations [, there being some notable exceptions, particularly for Weibull alternatives; citations omitted]. One generally must look at simulated powers for two or three sample sizes with a limited set of alternatives and extrapolate to the desired sample size and alternatives. This is often difficult to do (1984, p. 1650).

The same year as Spurrier's overview marked the appearance of a chapter, "Tests for exponentiality," in the *Handbook of statistics*, offering a more structured and nuanced survey of the field (Doksum & Yandell, 1984).

Epps and Pulley (1986) proposed a test for exponentiality, as against alternatives with monotone failure, or 'hazard', rate, (MHR). Such alternatives are either IFR or DFR, with the provision that the rate is strictly monotonic on some open interval, thus, either IHR or DHR. The authors' "C-test" relies upon the sample characteristic function. Given a "suitable estimator" of the parameter vector, $\hat{\theta}$, $C\left(\hat{\theta}\right)=\int_{-\infty}^{\infty} D\left\{c_n\left(t\right), c_0\left(t;\, \hat{\theta}\right)\right\} dW\left(t\right)$ defines a class of test statistics, where $D\{\cdot\}$ is a distance function and the characteristic functions, respectively, of the sample and population are $c_n\left(t\right)=\dfrac{\sum_{j=1}^{n}\exp\left(itx_j\right)}{n}$ and $c_0\left(t\right)=\int_{-\infty}^{\infty}\exp\left(itx\right)dF_0\left(x;\theta\right)$ (Epps & Pulley, 1986, pp. 206-207).

Several sources were suggested for specification of the C-statistic. Asymptotic properties are investigated and critical values, based on transformed variables and analytical results, are tabulated by the Monte Carlo method with

10,000 replications for several critical percentiles and sample sizes up to 200. Power for the C test and four others is computed, showing number of rejections from 1,000 Monte Carlo trials at $\alpha = .05$, for sample size 20. The competing tests were Gail and Gastwirth's G test, Bickel and Doksum's Weibull-optimal S_3^* test, Hollander and Proschan's NBU test, and Hahn and Shapiro's WE_0 test (citations omitted). The alternatives included Weibull, gamma, modified extreme value, uniform, truncated normal, and shifted Pareto distributions, with various selections of parameter. In summary, the authors observed that

> the powers of C and G are similar and usually greater than those of the other tests. However, S_3^* is often slightly more powerful against DHR alternatives. For the gamma distribution and for Weibull alternatives with shape parameter known and greater than 1.0 there exist most powerful tests of exponentiality [citations omitted], and our simulations indicate that the powers of the C and G tests are quite close to these upper bounds (Epps & Pulley, 1986, pp. 208-211).

Rayner and Best (1986) modified "smooth tests" of Neyman, utilizing orthonormal functions and based on the quadratic score statistic, to obtain omnibus, goodness-of-fit tests, including a test for exponentiality among others. The orthonormal system for the exponentiality test uses Laguerre polynomials, $L_r(x) = \sum_{s=0}^{r} \binom{r}{s} \frac{(-x)^s}{s!}$. The components, $\hat{V}_i = \frac{\sum_{j=1}^{n} L_i\left(\frac{X_j}{\bar{X}}\right)}{\sqrt{n}}$, contribute to the test statistic, $S_k^2 = \sum_{i=1}^{k+1} \hat{V}_i^2$. The authors prepared their results, in part, to compare with those of Angus's (1982) review of goodness-of-fit tests for exponentiality. The test statistic, S_k^2, is asymptotically distributed as Chi-square with k degrees of freedom. The authors found a reasonable rate of convergence, with no more than one percent error using critical values adjusted for sample size, for n $\geq$ 10 (Rayner & Best, 1986, p. 442).

Rayner and Best affirmed the exemplary performance of Gail and Gastwirth's Gini statistic as an omnibus test, despite the fact that "no omnibus test is always most powerful," and therefore use this statistic as a benchmark for their S_k^2 statistic (1986, pp. 442-443). In a Monte Carlo study, the authors found S_3^2 and S_4^2 test statistics to perform comparably well, in respect to a number of alternatives including Chi-square, lognormal, Weibull, beta, uniform, Pareto and shifted Pareto, and mixed Chi-square alternatives, with tabulated power varying with alternative, in a pattern similar to that for the Gini statistic. They conclude with an application of their statistic to a sample data set of operational lifetimes presented by Angus (1982). See Rayner and Best (1986, pp. 443-444).

Two additional articles of tangential interest were Rukhin and Strawderman (1982) and Samanta and Schwarz (1988). Rukhin and Strawderman investigated the estimation of a quantile of an exponential distribution with unknown location and scale parameters, with the objective of analytically determining maximal improvement of the estimator for the small sample sizes typical in fatigue testing such as that regulated by the Federal Aviation Administration (Rukhin & Strawderman, 1982, p. 162). Samanta and Schwarz modified the Shapiro-Wilk test for two-parameter exponentiality for the case when a number of smallest, and another number of largest observations have been censored. The authors' Monte Carlo study showed power against various alternatives, for uncensored data, and for data with numbers of observations, to wit, 2, 3, 5, or 10, censored from the low end, high end, or both ends of the sample (Samanta & Schwarz, 1988, pp. 529-530).

The 1990s.

In a concise treatment, outstanding from a pragmatic perspective, Ascher (1990) surveyed fifteen tests for exponentiality and compares their performance simultaneously, in one table, with fixed n = 20 and 1,000 Monte Carlo trials for each combination of test statistic and alternative distribution. Alternatives were classified as monotonic increasing hazard rate, monotonic decreasing hazard rate, non-monotonic hazard rate, and contaminated exponential distributions, the latter for the purpose of determining sensitivity to outliers (1990, pp. 1819-1821). Ascher corroborates that it is advantageous, in selecting a procedure to test exponentiality, to have specific information about the alternative distribution. Tests which performed well without such information included those of Cox and Oakes, Deshpande, Lorenz, Gnedenko, Hollander and Proschan, Lin and Mudholkar, Petra, Gini, Kolmogorov-Smirnov, and Hahn and Shapiro (citations omitted, Ascher, 1990, p. 1822).

LaRiccia (1991) developed a test for exponentiality analogous to Neyman's smooth tests for goodness of fit, based upon the quantile function.

Regression techniques are used to compute estimates of coefficients for the test statistic, which has an asymptotically Chi-square distribution (1991, pp. 427-429). Particularly as against Chi-square and lognormal alternatives, the author's T_k statistic performed better than the Gini statistic, which achieves power at least as high as T_k as against Weibull and Beta alternatives in almost all such cases shown in the author's Monte Carlo study. See, also, Kallenberg and Ledwina (1997), investigating, through simulation, the power of data-driven smooth tests for exponentiality and normality.

Ebrahimi, Habibullah, and Soofi (1992) presented a test for the exponential distribution, based upon estimation of the discrimination information between the empirical distribution and the hypothesized, one-parameter exponential distribution, whether or not the parameter is specified. The Kullback-Leibler (K-L) information function was used to discriminate between the two distributions (see also Kullback and Liebler, 1951; Kullback, 1968). The K-L information, distinct from Fisher information, is based on the concept of entropy, fundamental to information theory as originated by Claude Shannon and Norbert Weiner and developed by others, beginning around the middle of the Twentieth Century. This field, originally called communication theory, dealt with uncertainty in the transmission of data in such media as telegraphy (see Ash, 1965; Kullback, 1968; Soofi, 1994, 2000; Soofi, Ebrahimi, & Habibullah, 1995). Tests for goodness-of-fit based on entropy have demonstrated high power for null hypotheses of normality (Arizono & Ohta, 1989; Vasicek, 1976) and uniformity (Dudewicz & van der Meulen, 1981).

As presented by the authors, the K-L information is given by

$$I\left(F:F_0\right)=\int_0^\infty f\left(x\right)\ln\left\{\frac{f\left(x\right)}{f_0\left(x\right)}\right\}dx\,, \tag{2.17}$$

where the null distribution is $f_0\left(x;\lambda\right) = \lambda\exp\left(-\lambda x\right)$. $I\left(F:F_0\right)$ is known to be nonnegative and to vanish if and only if the density functions, $f\left(x\right)$ and $f_0\left(x\right)$ are equal almost everywhere, in the measure-theoretical sense. The authors developed a discrimination information statistic,

$I\left(F:F_0\right) = -H\left(F\right) - \ln\lambda + \lambda\int_0^\infty x\,f\left(x\right)dx = -H\left(F\right) - \ln\lambda + 1$, where $H\left(F\right)$ is the entropy of the distribution, F. This entropy is given by

$H\left(F\right) = -\int_0^\infty f\left(x\right)\ln\left[f\left(x\right)\right]dx$. Although entropy in general is difficult to calculate, several estimators have been provided. The authors used Vasicek's estimator, based on order statistics, $X_{(i)}$ from a random sample, size n, given by

$$H_{mn} = \frac{1}{n}\sum_{i=1}^{n}\ln\left\{\frac{n}{2m}\left(X_{(i+m)}-X_{(i-m)}\right)\right\}, \tag{2.18}$$

where the window size, m, is a positive integer less than $\frac{n}{2}$ and where $X_{(j)}$ is taken as equal to the first or nth order statistic in the case j < 1 or j > n, respectively (Ebrahimi et al., 1992, pp. 739-740; and see Vasicek, 1976).

For the case of unspecified parameter for the null distribution, using the sample mean providing $\hat{\lambda} = \frac{1}{\bar{x}}$, the K-L information between the empirical and null distributions, $I\left(F : F_0\right)$ is estimated by $I_{mn} = -H_{mn} + \ln\left(\bar{x}\right) + 1$. “Large values of I_{mn} indicate that the sample is from a non-exponential distribution” (1992, p. 741). The authors proceeded to a normalizing transformation, yielding a left-tailed test, for the test statistic, with values on the open interval, (0, 1), given by $KL_{mn} = \exp\left(-I_{mn}\right) = \exp\left(H_{mn} - \ln\left(\bar{x}\right) - 1\right) = \frac{\exp\left(H_{mn}\right)}{\exp\left(\ln\left(\bar{x}\right) + 1\right)}$. In the case that the exponential parameter is specified as $\lambda = \lambda_0$ in the null hypothesis, the normalized test statistic is given by $KL_{mn}\left(\lambda_0\right) = \exp\left(H_{mn} - \ln\left(\lambda_0\right) - 1\right)$ (Ebrahimi, et al., 1992, p. 741).

For the purposes of their Monte Carlo study, the authors set the window size, varying with sample size. Power comparisons include the Van-Soest and Finkelstein and Schafer’s statistics, W^2 and S^* respectively, with Weibull, gamma, and log-normal alternatives. The tabulated results show the superiority of the K-L information function, as implemented by the authors, in virtually all of the cases tested (Ebrahimi, et al., 1992, pp. 744-747).

The multitude of published papers on testing exponentiality sometimes evinces an at least temporarily unrecognized equivalence of reported work. Metz, Haccou, and Meelis (1994) noted the distinction of the Shapiro-Wilk test in dealing with the situation in reliability testing of a dead time, during which no events occur, after which a constant hazard rate appears. They also note that Darling's test (right-sided) is "locally most powerful against mixtures of exponentials." The authors demonstrate that these two tests are equivalent, by means of a simple transformation, $y_{(i)} = x_{(i)} - x_{(1)}$. Although the test statistics are asymptotically normal in distribution, convergence is so slow as to achieve acceptability around n = 500. The authors therefore extend Shapiro and Wilk's Monte Carlo table of critical values for values of n up to 500 (Metz et al., 1994, pp. 527-529).

Hollander and Proschan (1975; see also Hollander & Proschan, 1976; Hollander & Proschan, 1980) derived a V^* statistic based on a linear function of order statistics, for testing exponentiality against alternatives with decreasing mean residual life and expand a total-time-on-test procedure, beyond the IFR class, to the larger, NBU class of alternatives (1975, pp. 585-586). Ahmad (1992), referring to Hollander and Proschan (1975), proposed a new U-statistic for testing exponentiality against decreasing mean residual life, stating the advantages of easier computation and higher Pitman ARE, with linear failure rate, Makeham, and Weibull alternatives (Ahmad, 1992, pp. 416-418). Kumazawa (1993) finds the latter comparison inappropriate, however, on showing the Ahmad and Hollander and Proschan statistics, for testing for exponentiality against decreasing mean residual life, to be equivalent (1993, p. 474).

Basu and Harris (1994) investigated, analytically and with Monte Carlo simulations, the robust prediction of exponential distributions based on the Hellinger distance estimator, applied within the context of bootstrap methods. The K-L information, or divergence, is utilized to evaluate the 'best' predictive distribution. The authors present their procedure as an improvement over maximum likelihood estimation and work in a Bayesian context (1994, pp. 790-793).

Xu and Yang (1995), building on work of Seshadri et al. (1969) and others (citations omitted), investigated a characterization of the exponential distribution by a vector function of Type II censored data. Letting $X_{(i)}$ be order statistics of n independent, identically distributed (iid), nonnegative random variables with n > 2, $X_{(0)}$ set equal to 0, and the number of observations, r, between 2 and n inclusive, the authors define $S_{i,n} = \sum_{j=1}^{i}(n-j+1)\left(X_{(j)} - X_{(j-1)}\right)$ for i = 1 to n. Then let $W_{r,n} = \left(\frac{S_{1,n}}{S_{r,n}}, \frac{S_{2,n}}{S_{r,n}}, \ldots, \frac{S_{r-1,n}}{S_{r,n}}\right)$, a vector. They proceed to prove, under expanded conditions, a conjecture of Dufour (citation omitted) that, if $W_{r,n}$ is distributed as a random vector of order statistics of r – 1 from the uniform distribution on the interval, (0, 1), then the $X_{(i)}$ are exponentially distributed and that the converse is true, i.e., the conjecture characterizes exponentiality. In particular, the authors show that the conjecture is true in general for $n \geq r \geq 5$ and, if it is known that the given iid variables are either NBU or NWU, that the conjecture is true when both $n \geq r \geq 2$ and $n \geq 3$ (Xu & Yang, 1995, p. 769).

Gupta and Richards (1997) demonstrated analytically that some statistics classically used for testing exponentiality can be shown to have the same distributional characteristics, if the data, rather than being taken as a sample of iid variables, are viewed as coming from a multivariate Liouville density function. Such data could arise, in a reliability setting, when “components in a parallel-redundant system are subject to common environmental stresses” (1997, p. 204).

Chaudhuri (1997) proposed a test statistic for testing for exponentiality, against alternatives from the L-class of life distributions. An absolutely continuous life distribution, F, belongs to the L-class if, for y ,

$\int_0^\infty \exp(-st)\bar{F}(t)\,dt \;\geq\; \mu(1+s\mu)^{-1}$. The expression on the right of the inequality is the Laplace transform of the exponential distribution with mean, μ. In a hierarchy of life distributions, the L-class strictly contains the “harmonically new better than used in expectation” class (HNBUE), and ”hence … also contains the smaller NBUE, NBU, IFRA and IFR classes of life distributions” (1997, p. 249). Analytically, the author’s test statistic constitutes the supremum, or least upper bound, of the integrated difference between the left and right sides of the foregoing inequality with the empirical distribution in place of F, where equality reflects the no aging property and characterizes an exponential distribution (1997, p. 250). A computational formula is provided, given a fixed ε, $0 < \varepsilon < 1$:

$T_n = \frac{\sqrt{n}}{\bar{X}_n} \cdot \max_{1 \le j \le m} \left\{ \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) a_{ij} - \frac{\bar{X}_n}{1 + \frac{j}{m} F^{-1}(1-\varepsilon)\, \bar{X}_n} \right\}$, where $m = n(1-\varepsilon)$ and

$a_{ij} = \frac{\exp\left[\left(-\frac{j}{m}\right) F^{-1}(1-\varepsilon)\, X_{(i)}\right] - \exp\left[\left(-\frac{j}{m}\right) F^{-1}(1-\varepsilon)\, X_{(i+1)}\right]}{\left(\frac{j}{m}\right) F^{-1}(1-\varepsilon)}$. Approximate

critical values of T_n can be obtained from the asymptotic distribution, which associates T_n with Z_ε, where $P\left(\sup_{0 \le p \le 1-\varepsilon} Z_\varepsilon(p) > c \right) \le 2P\left(X > \frac{c}{K^{1/2}(1-\varepsilon, 1-\varepsilon)} \right)$,

where X has the standard normal distribution and

$K^{-1}(1-\varepsilon, 1-\varepsilon) = \left[\mu(1-\varepsilon)+1\right]^2 \left[2\mu(1-\varepsilon)+1\right]$, where the last factor on the right is here corrected, based upon the author's definition of the covariance function, $K(p,q)$ by the formula, $K^{-1}(p,q) = (p\mu+1)(q\mu+1)(p\mu+q\mu+1)$ (Chaudhuri, 1997, pp. 250-251).

The work of Hollander and Proschan and their collaborators influenced the direction of research on life distributions in the 1990's. Continuing their work framed under a taxonomy of life distributions, Hollander, Park, and Proschan (1986), introduced the NBU-t_0 and NWU-t_0 classes of life distributions, "new better than used of age t_0" and "new worse than used of age t_0", respectively. These are defined by the condition that the survival probability at age zero is, respectively, greater than or equal to, or less than or equal to, the survival probability at a specified positive age, t_0. The authors provided examples in areas of cancer diagnosis and survival, overhaul of commercial airplane engines required by the FAA, and the analog of “infant mortality” in manufactured components (Hollander et al., 1986, p. 91). A test based upon a U-statistic was proposed and ARE was studied, relative to the statistic of Hollander and Proschan (1972) and linear failure rate, Makeham, and a member of the class of NBU-t_0 distributions (Hollander et al., 1986, pp. 92-94).

Guess, Hollander, and Proschan (1986) extended the taxonomy further to two additional, new, nonparametric classes, to wit, “increasing initially, then decreasing mean residual life” (IDMRL) and “decreasing, then increasing mean residual life” (DIMRL), representing, respectively, aging that is “initially beneficial, then adverse” and “aging that is initially adverse, then beneficial” (1986, p. 1388). Both classes entail a turning point, τ , which may, in alternative procedures, be know or unknown respectively. In the case τ is unknown, $\rho = F(\tau)$, the proportion of the population which does not survive the turning point, may be known, motivating the authors’ second test (1986, p. 1389). High human mortality during infancy and attrition in initially intense training programs are examples of IDMRL distributions which respectively may satisfy the two cases pertaining to the turning point.

Hawkins, Kochar, and Loader (1992), addressing an open problem noted by Guess et al., proposed two tests for exponentiality, against IDMRL alternatives, which do not require knowledge of either τ or ρ (1992, p. 281). The authors’ Monte Carlo study showed that their second test dominated the first and compared well with the tests of Guess et al., when the turning point occurs below the 75th percentile of the life distribution, but not above (Hawkins et al., 1992, pp. 281-282). In addition to the theoretical development, computational formulae are provided which, in the case of the second test statistic, requires recursive computation, for which a FORTRAN program may be obtained from the first author (1992, pp. 282-283).

Ahmad (1998), following the work of Hollander et al. (1986) and Ebrahimi and Habibullah (1990) on life distributions which are new better than used of specified age (NBU-t_0), proposed a test focusing on a value of t_0 which is unknown but estimated, as either a specified percentile or the mean of the distribution. A NBU-t_0 distribution is defined analytically by the relation, $\bar{F}(x+t_0) \leq \bar{F}(x)\bar{F}(t_0)$ for all $x \geq 0$, where, as usual, $\bar{F}(x) \equiv 1 - F(x)$, the survival function (Hollander et al., 1986, p. 91). The NBU-t_0 class "is much larger than" the NBU class, which the former contains. Reversal of the inequality yields the "new worse than used of age t_0" class (Hollander et al., 1986, p. 91).

Ebrahimi and Habbibullah (1990) proposed a new test, based upon a weaker property than that defining the NBU-t_0 class, to wit, $\bar{F}(x+kt_0) \leq \bar{F}(x)\left\{\bar{F}(t_0)\right\}^k$ for all $x \geq 0$, where k is a positive integer, calling the class of distributions defined by this weaker property "new better than used of order kt_0" (or correspondingly, "new worse than used of order kt_0"). After a theoretical development of the defining integrals, Ebrahimi and Habbibullah presented their test statistics in terms of sample data, $T_k = T_{1k} - T_{2k}$, where

$T_{1k} = \left\{ \frac{1}{\binom{n}{2}} \right\} \cdot \sum_{1 \le i \le j \le n} \Psi\left(X_i, X_j + kt_0\right)$ and $T_{2k} = \left\{ \frac{1}{2\binom{n}{2}} \right\} \cdot \sum \Psi_k\left(X_{i_1}, \ldots, X_{i_k}\right)$, where the second sum "is over all combinations of k integers $\left(i_1, \ldots, i_k\right)$ chosen out of $\left(1, \ldots, n\right)$; and where $\Psi\left(a, b\right) = \begin{cases} 1 & \text{for } a > b \\ 0 & \text{otherwise} \end{cases}$ and $\Psi_k\left(a_1, \ldots, a_k\right) = \begin{cases} 1 & \text{for } \min_{1 \le i \le k} a_i > t_0 \\ 0 & \text{otherwise} \end{cases}$.

T_{1k} and T_{2k} are U-statistics with kernels, Ψ and Ψ_k of degree 2 and k respectively. T_1 reduces to the test statistic of Hollander et al. (1986). See Ebrahimi and Habbibullah (1990, pp. 212-213). The authors proceeded to study the ARE of their statistic, relative to Hollander, Park, and Proschan's (1986) statistic.

Ahmad (1998) first showed that the test of Ebrahimi and Habbibullah (1990) continues to hold, and reduces to simpler form, where $t_0 = \xi_p$, the pth percentile, estimated from the data. In this case, the statistic $\hat{T}_k$ is used to test the null hypothesis of equality, in the defining inequality for the NBU-t_0 class versus the alternative that an empirical distribution comes from NBU of specified age, ξ_p, where ξ_p is estimated by the order statistic, $X_{([np])}$ where [np] indicates the largest integer less than or equal to np. $\hat{T}_k$ is given by

$$\hat{T}_k = \frac{1}{2}\left(1 - p\right)^k - \binom{n}{2}^{-1} \cdot \sum_{1 \le i < j \le n} I\left(X_i > X_j + kX_{([np])}\right)$$

(Ahmad, 1998, pp. 451-452). Next, Ahmad particularized the Ebrahimi-Habbibullah test for testing NBU at mean age, i.e., $t_0 = \mu$, with μ estimated by $\overline{X}$. Here, the test statistic, $\hat{T}_k = \frac{1}{2}\hat{T}_{1k} - \hat{T}_{2k}$, where

$\hat{T}_{1k} = \binom{n}{k}^{-1} \cdot \sum_{1 \le i < \ldots i_k \le n} I\left\{\min\left(X_{i_1}, \ldots, X_{i_k}\right) > \overline{X}\right\}$ and $\hat{T}_{1k} = \binom{n}{k}^{-1} \sum_{1 \le i < j \le n} I\left(X_i > X_j + k\overline{X}\right)$,

where I represents the indicator function, yielding unity when the condition is satisfied and zero otherwise (Ahmad, 1998, pp. 452-454). Ahmad remarked that the first test is distribution free, although the second, as well as those of Hollander et al. (1986) and Ebrahimi and Habbibullah (1990) are not (1998, p. 455).

Ahmad and Alwasel (1999) proposed a goodness-of-fit test for exponentiality based upon the memoryless property, which characterizes the exponential distribution, expressed in the form, X is exponential with parameter μ if and only if $\overline{F}(2x) = \overline{F}^2(x)$ for all $x \ge 0$ (1999, p. 681). The authors developed a generic test statistic, resolving their study upon selected values or ranges for two constants refining the attributes of the statistic. The resulting statistic is $\hat{\Delta}_2\left(F_{n,\gamma}\right) = \frac{1}{n}\sum_{i=1}^{n} \overline{F}_n\left(2X_{(i)}\right) - \frac{2}{n}\sum_{i=1}^{n} \overline{F}_n^2\left(2X_{(i)}\right)\overline{F}_n^2\left(X_{(i)}\right) -$

$$\frac{2\gamma}{n}\left\{\sum_{i \text{ is odd}} \overline{F}_n\left(2X_{(i)}\right)\overline{F}_n^2\left(X_{(i)}\right) - \sum_{i \text{ is even}} \overline{F}_n\left(2X_{(i)}\right)\overline{F}_n^2\left(X_{(i)}\right)\right\} + o_p\left(n^{-1}\right).$$

The authors found values of γ in the interval, [0.3, 0.8] to be satisfactory, larger values yielding a higher powered test, with power approaching or equal to unity available for most cases presented. The alternatives included in the Monte Carlo power study were Weibull, gamma, and log-normal distributions. Sample sizes 10 and 20 were used. (1999, pp. 685-686). Critical values determined by Monte Carlo simulation evince the asymptotical relationship of the test statistic to the standard normal distribution, expressed by the authors as $\frac{n^{\frac{1}{2}}\hat{\Delta}_2\left(F_{n,\gamma}\right)}{\sigma_0\left(\gamma\right)} = Z_\gamma$. The authors recommended using this relationship, along with Monte Carlo methods, to determine appropriate critical values of γ, such that $Z_\gamma > z_{1-\alpha/2}$ (1999, p. 686).

The 2000s.

Ahmad proposed a test for exponentiality against common positively aging life distribution alternatives, using kernel methods of curve fitting for estimating probability density, with positive results in terms of ARE and power, studied by Monte Carlo methods (2000, p. 244). Ahmad focused on “the smallest class preserved under the formation of multicomponent systems,” to wit, increasing failure rate average (IFRA) distributions, which are defined by the condition that $-\ln\left(\bar{F}\left(x\right)\right)/x$ is nondecreasing for x ≥ 0 (Ahmad, 2000, p. 244), where $\bar{F}\left(x\right)$ represents the survival function, 1 – F(x) for a nonnegative random variable, X.

The departure of the data from a null hypothesis of exponentiality, in favor of an IFRA distribution which is not exponential may be measured in respect to a lemma proved by the author, for an IFRA pdf, f(x), holding that

$\delta_F = \int_0^\infty x \cdot f^2(x)\,dx \geq \frac{1}{4}$, with equality being consistent with the null hypothesis of an exponential distribution, as against IFRA and not exponential. Ahmad proposed estimation of δ_F by $\hat{\delta}_F = \left[n(n-1)a_n\right]^{-1} \sum_{i \neq j} \sum X_i k\left(\frac{X_i - X_j}{a}\right)$, where k is a known pdf, with finite variance and symmetric around zero, for which the author stated that the standard normal will suffice. A sequence a_n was selected according to the requirements that $na_n \to \infty$ and $na_n^4 \to 0$, so that $a_n = cn^{-1/m}$ will suffice for m = 2, 3, or 4, for constant, c . Having proved the null variance of this statistic to be $\frac{5}{108}$, the author provided $\frac{\sqrt{n}\left(\hat{\delta}_F - \frac{1}{4}\right)}{\sqrt{\frac{5}{108}}} > z_\alpha$, where z_α is the standard normal critical value for significance level α, as the criterion for rejecting H_0 (Ahmad, 2000, pp. 245-250).

Ahmad tabulated critical values of δ_F, studied relative efficiency, comparing with Weibull, linear failure rate, and Makeham alternatives, and found relative asymptotic efficiencies greater than unity in almost all cases, in respect to several other test statistics (2000, pp. 252-253). It was also claimed that the author’s test achieved higher power against the Weibull and other alternatives than Gail and Gastwirth’s tests based on the Lorenz curve and Gini index (2000, p 256).

Sen and Srivastava, addressing the problem in reliability engineering of minimizing costs in the replacement of units in active use, used U-statistics to test for exponentiality, against classes of life distributions pertinent to replacement theory (2000, p. 157). Two such classes, investigated in earlier studies, the New Better than Used and New Better than Used in Expectation classes, respectively, are limited to the case of positive aging throughout the lifespan of a tested unit, whereas “we encounter many situations in real life where there exists a particular age t_0 … at … which deterioration sets in.” Distribution classes devised for such cases include:

1. the New Better than Used with respect to the Set [t_0, ∞) class, denoted NBU-[t_0, ∞), defined by the existence of $t_0 > 0$ for a life distribution, F, such that $\bar{F}(x+t) \le \bar{F}(x)\bar{F}(t)$ for all $x \ge 0$ and $t \ge t_0$;

2. the New Better than Used of Age t_0 class, denoted NBU-{t_0}, defined by the existence of $t_0 > 0$ for a life distribution, F, such that $\bar{F}(x+t_0) \le \bar{F}(x)\bar{F}(t_0)$ for all $x \ge 0$;

3. the New Better than Used of Order kt_0 class, for k = 1, 2, …, denoted NBU-{kt_0}, defined by the existence of $t_0 > 0$ for a life distribution, F, such that $\bar{F}(x+kt_0) \le \bar{F}(x)\{\bar{F}(t_0)\}^k$ for all $x \ge 0$ [if k = 1, this class reduces to that above];

4. the New Better than Used in Expectation with respect to the Set [t_0, ∞) class, denoted NBUE-[t_0, ∞), alternatively the New Better than Old in

Expectation (NBOE), defined by the existence of $t_0 > 0$ for a life distribution, F, such that $\int_t^\infty \bar{F}(u)\,du \le \mu_F \bar{F}(t)$ for all $t \ge t_0$ with strict inequality for some t; and

5. the New Better than Used in Expectation of Age t_0 class, denoted NBUE-$\{t_0\}$, alternatively the New Better than Some Used in Expectation (NBSUE), defined by the existence of $t_0 > 0$ for a life distribution, F, such that $\int_{t_0}^\infty \bar{F}(u)\,du < \bar{F}(t_0)$ (Sen & Srivastava, 2000, pp.158-160).

The foregoing classes permit consideration of reliability studies involving a bath-tub shaped failure distribution, in which, for example, an initial burn-out period occurs, during which defective items that fail early are removed, leaving a residual population that ages gradually, with increasing failure rate. The authors proceeded to discuss tests for such situations in the literature (2000, pp. 60-61). The focus of their paper was on testing a null hypothesis of exponentiality (no aging) against two alternatives, respectively, H_1: F is NBOE and not exponential and H_2: F is NBSUE and not exponential. U-statistics were developed for these tests, asymptotic properties were treated, Pitman's ARE were studied, and Monte Carlo power comparisons with other tests were reported (2000, p. 162).

The test statistics involved a sequence of steps, beginning with the definition of a logical function, I(a>b) which equals 1 if a>b and 0, if not. The test statistics for the NBOE and NBSUE alternatives were W_n and U_n, respectively. For W_n, the authors defined the kernels ϕ_1 and ϕ_2, respectively, by $\phi_1(X_1, X_2, X_3) = X_1 \cdot I(X_2 > t_0) \cdot I(X_3 > t_0)$ and $\phi_2(X_1, X_2) = (X_1 - X_2) \cdot I(X_2 > t_0)$.

Then the composites W_1 and W_2 were defined, involving sums over the permutations of three or two subscripts: $W_1 = \frac{\sum_{\substack{all\ permutations\ of \\ 1,\ldots,n,\ taken\ 3\ at\ a\ time}} \phi_1\left(X_{i_1}, X_{i_2}, X_{i_3}\right)}{{}^nP_3}$ and $W_2 = \frac{\sum_{\substack{all\ permutations\ of \\ 1,\ldots,n,\ taken\ 2\ at\ a\ time}} \phi_2\left(X_{i_1}, X_{i_2}\right)}{{}^nP_2}$. Finally, the test statistic was defined as $W_n = \frac{W_1}{2} - W_2$. As the linear function of two U-statistics, W_n is itself a U-statistic. The authors remarked that W_n is not scale invariant.

For U_n, the authors defined the kernels δ_1 and δ_2, respectively, by $\delta_1\left(X_1, X_2\right) = X_1 \cdot I\left(X_2 > t_0\right)$ and $\delta_2\left(X_1\right) = \left(X_1 - t_0\right) \cdot I\left(X_1 > t_0\right)$. The composites U_1 and U_2 are then defined: $U_1 = \frac{\sum_{\substack{all\ permutations\ of \\ 1,\ldots,n,\ taken\ 2\ at\ a\ time}} \delta_1\left(X_{i_1}, X_{i_2}\right)}{{}^nP_2}$ and $U_2 = \frac{\sum_{i=1}^{n} \delta_2\left(X_i\right)}{n}$.

Finally, the test statistic is defined as $U_n = U_1 - U_2$.

Significantly large W_n are evidence that the distribution is NBOE and not exponential, whereas significantly small values are evidence of a NWOE (New Worse than Old in Expectation) or NWUE-$[t_0, \infty)$ alternative. Significantly large values of U_n indicate a NBSUE alternative while significantly small values indicate a NWSUE (New Worse than Some Used in Expectation) or NWUE-$\{t_0\}$ alternative to the exponential distribution (Sen & Srivastava, 2000, pp.163-165). Asymptotic normality of the statistics was presented, as well as large sample performance, in terms of ARE and also efficacy (See Gibbons & Chakraborti, 2003), with four alternative distributions, including the Makeham distribution, which reduce to the exponential distribution under H_0. The alternatives represent different distribution classes, among which the authors discussed the existing hierarchal relationships (Sen & Srivastava, 2000, p. 169).

Gupta and Ramanayake (2001) utilized Monte Carlo simulations with 10,000 repetitions to explore the power of three test statistics aimed at determining a change in linear trend, to wit, change in the parameter of an exponential distribution. The authors noted that, although much of the literature treats abrupt change, “it is more realistic to assume that the change occurs following some intervention over a period of time … known in the literature as the change point with linear trend”. Consequentially, the alternative hypothesis gave the parameter as a piecewise function of time, with an interval of linear change, (p,q], between initial and final intervals with the null parameter and a parameter with constant added to the null parameter, respectively (2001, p. 182). Two likelihood ratio-type statistics and Rao’s efficient score statistic were derived and compared, with applications.

Raqab and Ahsanullah (2001) explored the generalized exponential distribution (GE), introduced in 1999 by Gupta and Kundu (1999) and an incipient Twenty-First Century trend in the field. The pdf of the GE distribution is $f(x;\theta) = \theta\left(1-e^{-x}\right)^{\theta-1} e^{-x}$ for x > 0 and θ > 0. The cdf is given by

$F(x;\theta) = \left(1-e^{-x}\right)^{\theta}$. When the shape parameter, θ, equals one, the GE reduces to the ordinary exponential distribution with parameter equal to unity. When the parameter is a positive integer, the GE cdf "is the cdf of the maximum of a random sample of size θ from the standard exponential distribution." The authors give the median and moments for $\theta \geq 1$ as $-\ln\left(1-0.5^{\frac{1}{\theta}}\right)$ and $\mu^k = \theta k!\sum_{j=0}^{\infty}(-1)^j\binom{\theta-1}{j}\frac{1}{(j+k)^{k+1}}$. The GE distribution can represent either an increasing or a decreasing hazard rate, depending on the value of the shape parameter. (2001, pp. 109-110). The authors derive exact expressions for the means, variances and covariances of order statistics from a random sample from the GE distribution and obtain best linear unbiased estimators of the location and scale parameters, using Mathematica version 3.0 (2001, pp. 110-116).

Gupta and Kundu (2003b) discussed similarities and differences between the GE and Weibull distributions and explored minimum sample size for discriminating between them using a likelihood ratio test. Gupta and Kundu (2003a) treated similarities and differences between the GE and gamma distributions and approximation of a gamma distribution by a GE distribution. Gupta and Kundu (2004) continueed the authors' comparison of gamma and GE distributions, to find the minimum sample size for discriminating between the two, using a likelihood ratio test. Kundu, Gupta, and Manglick (2005) continued this trend of investigation, comparing the log-normal and GE distributions using the ratio of maximized likelihoods and determining minimum sample size for discriminating between them.

Another trend in recent research, tangential to the present focus, treats record values, a specialized topic in order statistics. Record statistics may be seen as “successive extremes in a sequence of” independent and identically distributed random variables (Raqab, 2001a; 2001b; see also Arnold, Balakrishnan, & Nagaraja, 1992; Awad & Raqab, 2000; David, 1970). Raqab (2002) derived exact means, variances, and covariances for record statistics of the GE distribution, utilizing special functions, including Riemann zeta and polygamma functions.

Scherb (2001) developed an algorithm (using Mathematica conventions) to determine uniformly most powerful, two-sided tests in a distribution-free fashion through an iterative process. This technique is applicable, not only for the one-parameter exponential families, but for any distribution family with a strictly monotonic likelihood ratio. The author distinguished the capacity of Mathematica to perform exact calculations, stating that, in the case of FORTRAN, using precision REAL*16, an appropriately small tolerance to test for a zero value is $\varepsilon = 10^{-28}$ (2001, pp. 71-72, 74-75).

Chi and Tsai addressed the dependence of comparative advantages of, in particular, weighted log-rank (WLR) and weighted Kaplan-Meier (WKM) tests upon the alternative distribution type, in testing for equality of two survival distributions, in the presence of right censorship. After comparing several tests in the literature, the authors proposed two new tests, the first using a linear combination of the WLR and WKM and the second using the maximum. The authors compared the performance of their tests through Monte Carlo power simulation and treated the asymptotic distributions (2001, pp. 743-745).

Klar (2001) presented an omnibus goodness-of-fit test for the exponential distribution (as well as one for the normal), based upon the integrated distribution function, $\Psi(t) = E(X-t)^{+} = \int_t^\infty (1-F(x))\,dx$ (2001, p. 338), which for the exponential cdf given as $H_0 : F(t,\theta) = 1-\exp(-\theta t)$ [13], yields $\Psi(t) = \dfrac{\exp(-\theta t)}{\theta}$. The empirical idf, $\Psi_n(t) = \int_t^\infty (1-F_n(x))\,dx = \frac{1}{n}\sum_{i=1}^{n}(X_i - t) I\{X_i > t\}$, where $I\{X_i > t\}$ is the indicator function and $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \le t\}$ is the empirical cdf, is compared with the estimated idf, $\Psi_n(t,\hat{\theta}_n) = \int_t^\infty \left(1-F_n(x,\hat{\theta}_n)\right)dx$, where $\hat{\theta}_n$ is taken equal to the reciprocal of the sample mean, the maximum likelihood estimator of θ. The test statistic, in one form derived by the author, was given as $T_n = \dfrac{1}{n}\sum_{i<j}\left(Y_{(i)}^2 Y_{(j)} - \dfrac{Y_{(i)}^3}{3}\right) + \dfrac{1}{3n}\sum_{i=1}^{n} Y_{(i)}^3 - 2\sum_{i=1}^{n}\left(e^{-Y_i} + Y_i - 1\right) + \dfrac{n}{2}$, where the $Y_{(i)}$ are order statistics of $Y_i = \hat{\theta} X_i$ (2001, pp. 339-340).

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[13] I.e., contrary to our convention for the exponential parameter, θ.

Critical values of T_n were computed by Monte Carlo method, using twenty percent trimmed means of 100 simulations, each with 10,000 replications (2001, p. 342). A power study was performed, with gamma, Weibull, log normal, uniform, half normal, half Cauchy, and Chi-square alternative distributions, with good results for an omnibus test. The C-vM and A-D and other tests in the literature were discussed (Klar, 2001, pp. 245-246).

Fortiana and Grané (2002) tested for exponential goodness-of-fit, proposing a statistic based on Hoeffding's maximum correlation. The authors explored the exact and asymptotic properties of the statistic and conducted power comparisons with the Gini and Shapiro-Wilk statistics and Stephen's modification of the latter (2002, p. 86).

Henze and Meintanis (2002) summarized a considerable amount of classical and contemporary research on goodness-of-fit tests for the exponential distribution. They proposed a test based upon a new characterization of the one-parameter exponential distribution (denoted Exp(θ)), based upon the characteristic function, given by $\Psi(t) = E\left[\exp(itX)\right] = u(t) + iv(t)$, for a nonnegative random variable, X. The authors presented, as a characterization of the exponential distribution,

> Theorem 1.1. Among all distributions of nonnegative random variables that possess a continuously differentiable density with finite limit as $x \to 0^+$ and an absolutely integrable derivative, the exponential law Exp(θ) is the only one that satisfies the equation $v(t) - \theta t u(t) = 0,\ t \in \Re$ (2002, p. 1480).

The proof utilizes an inversion formula for the Fourier sine transform, and is elaborated in one of the thirty references provided by the authors for this article.

The authors defined the empirical characteristic function for scaled data, $Y_j = \frac{X_j}{\bar{X}_n}$, to wit, $\phi_n(t) = \frac{1}{n}\sum_{j=1}^{n}\exp(itY_j) = C_n(t) + iS_n(t)$ and developed a test statistic, $W_n = n\int_0^{\infty}\left[S_n(t) - tC_n(t)\right]^2 w(t)\,dt$, where $C_n(t) = \frac{1}{n}\sum_{j=1}^{n}\cos(tY_j)$ and $S_n(t) = \frac{1}{n}\sum_{j=1}^{n}\sin(tY_j)$. The authors proceeded to develop two special cases of W_n, expressed algebraically as summations of sums of rational expressions and exponential functions of the scaled data, with two classes of weight functions (Henze & Meintanis, 2002, pp. 1481-1482).

After exploration of asymptotic behavior, the authors presented Monte Carlo studies, performed using double-precision FORTRAN and IMSL library subroutines when available. Comparison tests were those of Baringhaus and Henze; Henze; Epps and Pulley; C-vM; and tests “based on a characterization of exponentiality via the mean residual life function”. Alternative distributions considered specific members of the Weibull, gamma, lognormal, inverse Gaussian, half-normal, half-Cauchy, power, modified extreme value, linear increasing failure rate law, and shifted Pareto density families Extensive power tables and a concisely detailed summary of results are provided (Henze & Meintanis, 2002, pp. 1491-1495).

Taufer (2002) investigated goodness-of-fit tests for exponentiality based on the information-theoretic measure of entropy. The work of Gregorzewski and Wieczorkowski (1999) and Ebrahimi et al. (1992), and earlier work by Dudewicz and Van der Meulen (1981) and Vasicek (1976), utilizing the Kullbach-Leibler measure to discriminate between distributions, was extended by the author. The test employed uses the property that “for a density, f, positive on [0, ∞) and with finite mean μ, the entropy $H(f) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx$ is maximized for an exponential distribution with density [Equation 2.1]” (Taufer, 2002, p. 190). The author’s study compared statistics developed to test for exponentiality, with options for estimator and whether a transformation of data is used. Alternative distributions for power comparisons include members of the Chi-square, Pareto, half-normal, beta, gamma, Weibull, and lognormal families (Taufer, 2002, pp. 191-193).

The entropy-based tests use a transformation of a statistic giving the difference between Vasicek’s nonparametric estimator of entropy from sample data and the maximum likelihood estimator of entropy under null hypothesis, which in the case of exponentiality is given by $\hat{H}(f_0) = \hat{H}(f_E) = \log\left[\bar{X}e\right]$. Although two other estimators were discussed, the results of the investigation favor Vasicek’s estimator, which yields higher power “in nearly all cases”, leaving the issue of transformation of the data. Vasicek’s entropy estimator was given by

$H_{m,n} = \frac{1}{n}\sum_{i=1}^{n} \log \frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)$, where the $X_{(i)}$ are order statistics of the sample, $m < \frac{n}{2}$ is an integer, and $X_{(i-m)} = X_{(1)}$ if $m \geq i$ and $X_{(i+m)} = X_{(n)}$ if $i + m \geq n$ (Taufer, 2002, pp. 190, 198).

Seshadri et al. (1969) demonstrated a transformation of an exponential random sample of size n to an ordered sample of size n-1 from the uniform distribution on (0, 1). Taufer's "main aim is to compare the two types of entropy based tests", to wit, testing the original data or the transformed data. For this purpose, the alternatives are classified into IFR, DFR, and hump-shaped failure rate distributions. The transformations are denoted 'J' and 'K', yielding the variables Z_j and Z'_j, respectively, given as follows: $Z_j = \sum_{i=1}^{j} \frac{X_i}{S}$, where $S = \sum_{i=1}^{n} X_i$, and $Z'_j = \sum_{i=1}^{j} \frac{D_i}{S}$ where $D_i = (n-i+1)\left(X_{(i)} - X_{(i-1)}\right),\ i = 1, \ldots, n;\ X_{(0)} = 0$. Note that $S = \sum_{i=1}^{n} D_i$, as well (Taufer, 2002, p. 191).

Whereas Seshadri had good results using the foregoing transformations with Kolmogorov-type statistics, Taufer explored their use with an entropy test for uniformity, computing $H_{m,n-1}$ with either the n-1 Z_j or Z'_j, variable, with $m < \frac{n-1}{2}$. Critical values were published in Dudewicz & Van der Meulen (1981). The null hypothesis should be rejected for small values of the test statistic, in this case, $H_{m,n-1} - \hat{H}(f_E)$ (Taufer, 2002, pp. 191-192).

Both transformation and category of alternative had an effect, in Taufer's study. For the DFR distribution alternatives considered, including X_1^2 and Pareto with parameter equal to two or four, the tests after J or K transformation "obtain considerably higher power than the direct test even for small sample sizes." Although the J transformation outperformed the K transformation, particularly for m = 1, this is the only category for preferring J (Taufer, 2002, pp. 194-195). For both IFR distributions with decreasing density, including the half-normal, HN(0, 1), and beta with parameters one and two and IFR distributions with unimodal density, including gamma with parameters two and one and Weibull with parameter two, the test with J transformation "has negligible power for all cases", and the test on untransformed variables is generally superior to use of the K transformation. In this category, higher values of m achieve higher power (Taufer, 2002, pp. 195-196). For the hump-shaped failure-rate representatives here, the lognormal distributions derived from N(-.5, 1) and N(0, $\log \Gamma(3)$) respectively, the test with transformation K outperforms the other two test options, achieving highest power with larger values of m (Taufer, 2002, p. 196).

In sum, the author recommended using the J transformation only against DFR distributions, and not against omnibus alternatives. For omnibus testing, the K transformation is preferable as its power is usually higher than the test on untransformed data, except in the case of IFR decreasing density alternatives. If the latter type of alternative is specified, then the test without transformation should be used (Taufer, 2002, p. 198).

Taufer (2002, p. 190) briefly mentioned the work of Correa (1995), but did not treat the latter's spacings-based estimator of entropy in a comparative context. Correa furnished an interesting modification of Vasicek's (1976) entropy estimator: "[i]nstead of taking only two points to estimate the slope … we propose to use all the sample points in between to estimate β using a least squares estimator of the slope" (Correa, 1995, pp. 2441-2442). Also of interest is Gokhale's (1983) earlier implementation of the Kullbach-Liebler information, using Monte Carlo simulation for entropy-based goodness-of-fit testing, including for exponentiality. Gokhale discussed employment of the property of "entropy-uniqueness in certain families of densities" to derive critical quantiles of test statistics for goodness of fit (Gokhale, 1983, p. 157-158).

El-Bassiouny (2003) investigated the use of a moment inequality for the IFRA class of life distributions, to test a null hypothesis of exponentiality versus an IFRA alternative. The inequality, derived from a particular characterization of the IFRA class, to wit, $\bar{F}(kx) \le \left[\bar{F}(x)\right]^k$ for x > 0 and k > 1, is given as:

> Theorem 2.1. If F is IFRA then for all integers r $\geq$ 0, k $\geq$ 2, $\nu_{(r+1)} \geq \frac{\mu_{r+1}}{k^{r+1}}$, where $\nu_r = E\left[\min\left(X_1,...,X_k\right)\right]^r$, $\mu_r = E\left(X_1^r\right)$, and $X_1,...,X_k$ are independent identically distributed random variables with d.f. F (El-Bassiouny, 2003, pp. 445-447).

Fixing k = 2, the author proposed, ultimately, a scale invariant test statistic,

$\Delta_{r+1} = \dfrac{\nu_{(r+1)} - \dfrac{\mu_{r+1}}{2^{(r+1)}}}{\mu^{r+1}}$, which is estimated from the data by

$$\hat{\Delta}_{r+1} = \frac{\frac{2}{n(n-1)}\sum_{i<j}\left\{\min\left(X_i^{r+1}, X_j^{r+1}\right) - \frac{X_i^{r+1}}{2^{r+1}}\right\}}{\bar{X}^{(r+1)}}$$

, which, under the null hypothesis of exponentiality, has variance which can be approximated as 0.083 for r = 0 and as 0.573 for r = 1. Given asymptotic normality, "we reject H_0 if the calculated value of $\frac{\sqrt{n}\hat{\Delta}_{r+1}}{\sigma_0}$ exceeds the upper α -quantile of the standard normal distribution z_α" (El-Bassiouny, 2003, pp. 447-449). Based upon the foregoing test statistic, formulas for Pitman asymptotic efficacy, towards computation of Pitman asymptotic relative efficiency (PARE) were provided for Weibull, linear failure rate, and Makeham distribution family alternatives, with results of the author's Monte Carlo simulations (2003, pp. 449-453).

Gulati and Neus (2003) conducted a series of power studies, for goodness-of-fit testing for exponentiality versus several alternatives, where the available data is grouped.

CHAPTER 3

METHODOLOGY

Computing

Computers

Monte Carlo programs were composed and run on two personal computers, the first with an Intel Pentium 4 CPU running at 2.6 gigahertz and 512 megabytes of memory and the second with an AMD Athelon XP 2800 CPU (333 MHz FSB) running at 2.083 gigahertz and 512 megabytes of DDR memory. These computers were operating under the Microsoft Windows XP Professional and Windows XP Home Edition systems, respectively.

Programming

Programming of the specific procedures and utility subroutines for this Monte Carlo study was accomplished by the author using Essential Lahey FORTRAN 90 version 4.0 augmented with the IMSL FORTRAN 90 MP Library version 3.0.

Additional Software

Ancillary exploration, analysis, and presentation were performed using, in addition to FORTRAN programming, some widely available, commercially produced programs, to wit, Microsoft Excel[14], MINITAB[15], SPSS[16], and Maple[17]. DATAPLOT (Filliben & Heckert, 1978-present), a mathematical and statistical, FORTRAN-based package functioning as an interpreter for a defined set of user commands, described as “a free, public-domain ... software system for scientific visualization, statistical analysis, and non-linear modeling”[18] was also used, specifically for the simulation of data from the Makeham distribution. Handheld calculators were also used, incidental to the planning and analysis of the project and output.

Random samples were produced during execution, via ad hoc programming in FORTRAN and by subroutine calls to RANGEN (Blair, 1987; and see Headrick & Beasley, 2004), a FORTRAN module made available by the College of Education, Department of Educational Evaluation and Research, at Wayne State University, or prior to Monte Carlo execution, using the DATAPLOT package, MINITAB, and Excel, with storage in the form of FORTRAN formatted fixed-point constants, maintained in text files for access during execution, in the case of the Makeham distribution and two anomalous distributions (mixtures of

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[14] Copyright, Microsoft Corporation.

[15] Copyright, Minitab, Inc.

[16] Copyright, SPSS, Inc.

[17] Copyright, Waterloo Maple Inc.

[18] See http://www.itl.nist.bov/div898/software/dataplot/homepage.htm.

simulated data from Beta and Triangular distributions).

Preparation of the text was achieved using Microsoft Word, MathType version 5.2a, a product of Design Science, and Reference Manager Professional Edition version 10, a product of ISI ResearchSoft.

Background and Overview

Experimental Design

Gentle (2003) discussed the design and features of a variety of Monte Carlo experiments, confirming the importance of the Monte Carlo approach for modern statistical research. Such experiments may include the objectives of studying the robustness of a statistical procedure, with respect to Type I error in testing for a given distribution representing a null hypothesis, or with respect to Type II error over a class of alternative distributions (i.e., effects), or studying power, of a given procedure applied over a class of alternative effects, or more expansively, comparing power over a set of procedures, each applied over the same class of alternative effects. In such experiments, the treatments entail alternative distributions, alternative procedures – such as competing test statistics, or combinations of both. Additional variables may include sample size, density parameters for simulated random data samples representing different effects or alternative hypotheses, and significance levels for generating critical values in order to determine rejection rates (Gentle, 2003, pp. 297-311).

Sample Size and Number of Replications or Trials

Sample size is implicated in two ways in a Monte Carlo experiment, parallel to the two senses in which the term data may be employed. First, the number of replications is actually the experimental sample size – or number of trials – denoting the number of observations taken for the statistics under study, for each case or combination of factors. Second, there is the factor or controlled variable, sample size, which is an attribute of each of those observations, inasmuch as the latter are samples, to wit, computer simulated pseudo-random samples with an array of characteristics comprising the experimental design. The number of replications used in the present study was 100,000. The observations consisted of samples of size n = 3 to 100, narrowed for further treatment and analysis to four selected sizes, n = 4, 10, 20, and 50.

The Empirical Data

The data for the Monte Carlo experiment comes from what Mooney (1997) calls a pseudo-population of simulated values of a statistic, the behavior of which is under study (p. 67). The sample size is the number of replications, while different instances of the statistic or statistics studied may be generated from samples simulated with variable sample size, such size being a factor in the study. The latter samples partake in the controlled conditions of the experiment, whereas the observed statistics are the response variables for analysis.

Thus, the data, the observed statistics in replications enabled by the speed and capacity of contemporary electronic computers, indeed, by common personal computers, comprise very large samples (size 100,000 in the present investigation), which are suitable for empirical description and analysis akin to investigation of continuous random variables.

Data Analysis in a Monte Carlo Study

Analysis of the observations resulting from a Monte Carlo experiment often proceeds in lieu of or prior to mathematical analysis which would encounter difficulties including functional forms which are intractable or otherwise not readily accessible. The Monte Carlo analysis may consist in studying the results of series of algorithms which do not readily lend themselves to a theoretical approach under the regime of mathematical statistics. Mooney explains that a Monte Carlo simulation experiment may be used “when a statistic has no well-developed statistical theory regarding its distribution under any conditions. ... [when] a researcher may wish to use a statistic that seems to meet his or her substantive needs for a problem but about whose distribution little is known” (Mooney, 1997, p. 66; see also Gentle, 2003, p. 297).

Outline of the Study

The Theta-Star Goodness-of-Fit Method

The Theta-Star Estimate.

For the purpose of testing robustness in respect to Type I error, sampling was done from an exponential population. However, knowledge of the parameter is not assumed, and it must be estimated from the data. The Theta-Star method, which begins with approximation of the parameter under an assumption that a given random sample is exponentially distributed, is introduced. A distinguishing feature of the method herein is that both sample mean and standard deviation are utilized for estimation. In order to investigate the characteristics of a goodness-of-fit statistic, a critical value table was constructed, based upon repeated sampling, using pseudo-random samples from the exponential distribution with single parameter equal to unity.

Theta-Star refers to the result of an optimal estimate of the parameter, equilibrated between the sample mean and sample standard deviation, based upon the identity between the corresponding moments for a one-parameter exponential distribution. Despite the identity, there is considerable variation in the difference between the sample mean and sample standard deviation for samples taken from a given exponential distribution. This is exemplified by 2,000 samples, n = 25, from the exponential distribution with mean equal to 2.0, as simulated and displayed by MINITAB in figure 3.1. The Theta-Star estimate minimizes the sum of squared deviations between the data and estimated distribution (applied to the data by inverse cumulative transformation), for an estimator positioned linearly relative to the sample mean and standard deviation.

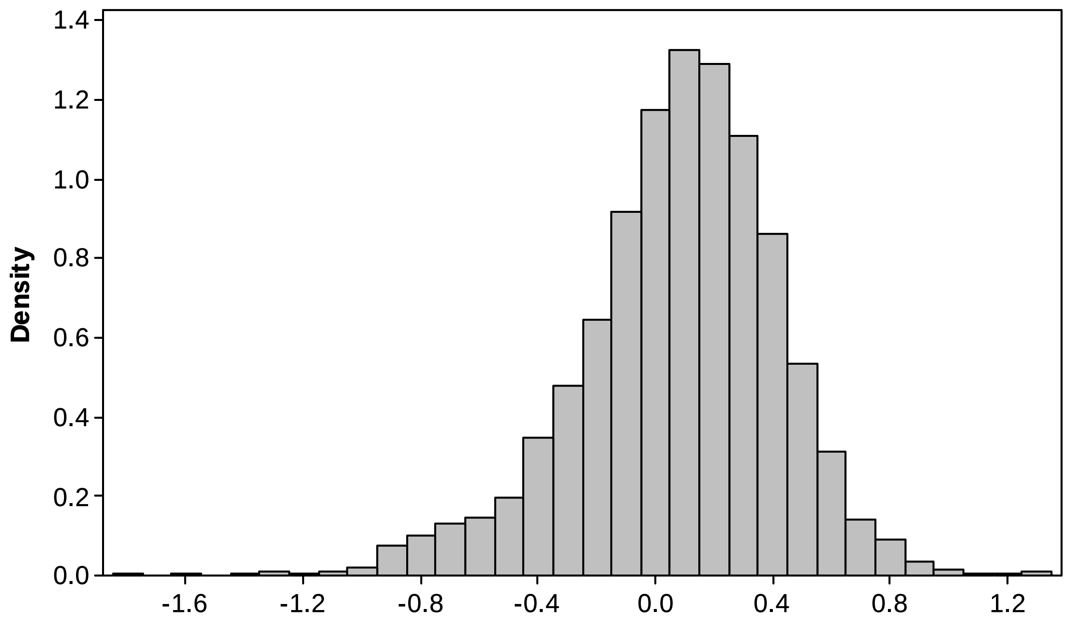

Figure 1. Histogram of the difference between sample mean and standard deviation for 2000 samples of size 25, randomly generated by MINITAB from the exponential distribution with mean = 2.0.

The test statistic, MADMASR.

After the estimated parameter, theta-star, was computed for a sample, an inverse exponential cumulative distribution function based on the estimate was applied to quantiles based upon the sample size (i.e., the empirical cdf), with the resulting variates stored in an array. Linear regression of this inverse cdf array on the sample was then performed and the residuals were stored in another array. Absolute values of the residuals were taken and then standardized and subsequently sorted. The sample by sample procedure and computations are implemented in subroutines located in the FORTRAN module, utils4tstexpdst, which is referenced by the FORTRAN programs directly employed in this study.

Continuing the procedure for each trial sample, for each sample size, the median absolute standardized residual was found. Another array was established, holding the absolute deviations of the absolute standardized residuals from their median value, and this array was sorted to obtain its maximum element. The Maximum Absolute Deviation, of the absolute standardized residuals from regression of the estimated distribution on the data, from the Median Absolute Standardized Residual was recorded, as MADMASR, the proposed goodness-of-fit statistic, first in an array, denominated ytestat in the FORTRAN program MCexpontest01, which outputs Table 2[19] described below. For each of the 42 sample sizes, n = 3 to 100, the 100,000 instances of MADMASR were stored in a file identified by sample size, in preparation for analysis of the sampling distribution.

The sampling distribution of the goodness-of-fit statistic, MADMASR.

The distribution of MADMASR for 100,000 replications is characterized by upper-tail critical values, at empirical cumulative probabilities, 0.90, 0.95, 0.975, and 0.99. This treatment is performed for sample sizes from 3 to 100, with n from 3 to 30 inclusive by increments of one and n from 35 to 100 by increments of 5.

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[19] Tables numbered by Roman numerals herein occur in the flow of the text and are employed to enhance readability. Tables numbered with Arabic numerals comprise the output of the researcher’s Monte Carlo computer programs. While each of the latter corresponds to a major stage in the Monte Carlo study and is introduced in relation to methodology in Chapter 3, its presentation is reserved for Chapter 4, where the results are treated.

The results are shown in Table 1[20].

Other empirical characteristics of the MADMASR statistic, as well as of Theta-Star, are summarized in Table 2[21]: for each sample size, the mean, standard deviation, skewness, and kurtosis[22] for all replications are tabulated.

Robustness and Power of the Theta-Star Method of Goodness-of-fit Testing

Robustness of MADMASR: Type I error rate.

In order to test the robustness of the goodness-of-fit statistic with respect to Type I error rate, critical values from Table 1 were saved for selected sample sizes, to wit, n = 4, 10, 20, and 50. New samples from the exponential distribution were simulated, with parameters equal to, respectively, 0.1, 0.5, 2 and 10. Using the stored critical values, 100,000 simulated random samples were taken for each of the four sample sizes and for each of the four parameter values. With null hypothesis based upon the exponential distribution with theta-star as its parameter, the rejections at each upper critical level were counted for Monte Carlo estimates of Type I error rates for the MADMASR statistic. The results are

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[20] Table 1 is an output of the FORTRAN program, MCexptst01_outproc.

[21] Table 2 is an output of the FORTRAN program, MCexpontest01.

[22] The sample skewness and kurtosis are computed, using the formula employed by Microsoft Excel 2003. The formulas are consistent with results obtained by the author using Minitab and SPSS as well, and are treated in Sokal and Rohlf (1995, pp. 114-115).

set out in Table 3[23], which also includes the mean and standard deviation over all replications of the theta-star estimates for each distribution and sample size.

Power of MADMASR: Rejection rates for non-exponential data.

The same sample sizes n = 4, 10, 20, and 50, with critical values from Table 1, were utilized to investigate robustness with respect to Type II error and power of the MADMASR test for exponentiality. Several alternative distributions were selected for Monte Carlo sampling, with 100,000 trials for each distributional source and sample size, for four upper critical values, resulting in a proportion of rejections for each case. The rejection rates, along with means and standard deviations for the parameter estimates from the simulated samples and average sample means and variances, are set out in Table 4[24].

The alternative probability distributions were selected from the following distributions: a mixture of two exponential distributions (with parameter equal to 0.5 and 4 respectively); a mixture of exponential (with parameter θ, and mean, equal to unity) and normal, N(10, 1), distributions (representing the combination of an early failure distribution during a burnout period and a later, bell-shaped wear-out period, in reliability analysis: see Bazovsky (1961, especially pp. 218-219); gamma distributions with shape parameter, λ equal to 1.6 and 4, respectively, and scale parameter, $\sigma = \frac{1}{\lambda}$, to achieve an expected value of unity

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[23] Table 3 is an output of the FORTRAN program, mcexp_verify.

(see Bury, 1999, pp. 208-209; Ebrahimi, Habibullah, & Soofi, 1992, pp. 745-746); Weibull distributions with shape parameter, λ, equal to 0.5, 1.5, and 2, respectively, and scale parameter, $\sigma = \dfrac{1}{\Gamma\left(1+\dfrac{1}{\lambda}\right)}$ (yielding 0.5, 1.10773, and 1.128387 respectively), to achieve an expected value of unity (see Bury, 1999, pp. 311-312; Ebrahimi, Habibullah, & Soofi, 1992, p. 745); the Gumbel (Type I extreme-value) distribution with location and scale parameters equal to 0.42278 and unity respectively (see Bury, 1999, p. 268; Lawless, 2003, p. 20); lognormal distributions, each with scale parameter of unity and with shape parameters of 0.2, unity, and 2 respectively (see Bury, 1999, pp. 154-155; Samanta & Schwarz, 1988; Schafer & Finkelstein, 1972); the Makeham distribution with shape parameters[25] ξ, λ, and θ each equal to unity (see Ahmad, Alwasel & Mugdadi, 2001; Barlow & Proschan, 1981, p. 229; Gnedenko, 1966, p. 74; Gnedenko et al., 1969, p. 12; Scollnik, 1995); Chi-Square distributions with degrees of freedom between one and four inclusive, the second of which reduces to the exponential with parameter equal to 2.0; the uniform distribution on the unit interval; the half-normal distribution (Gentle, 2003, p. 176; and see Barlow & Proschan 1981, treating the "'folded-over' normal distribution", p. 247); and two anomalously-

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[24] Table 4 is an output of the FORTRAN program, mcexp_rejectrates.

[25] Under the Digital Library of Mathematical Functions (DLMF) parameterization for the Makeham Distribution. See National Institute of Standards and Technology (2005), Chapter 8, "Probability Library Functions", MAKPDF.

shaped, finite populations (N = 1500 each) generated as mixtures of Beta and Triangular variates, respectively, the linear combination with coefficients equal to 0.5 of Beta with parameters 0.1 and 0.3 and Triangular on (0.4, 0.8) with mode at 0.5 and the combination of the product of a Bernoulli (probability 0.5) variate with the same Beta and the product of unity minus the Bernoulli variate with the same Triangular Distribution. See Bury (1999, p. 238) for treatment of the Beta Distribution and Evans, Hastings, and Peacock (1993, p. 149) for treatment of the Triangular Distribution, both of which were simulated using MINITAB.

Comparison With other Goodness-of-Fit Tests

Selection of the alternative tests.

Several alternative, competing tests for comparing an empirical (i.e. here, a pseudo-random sample) distribution with a hypothetical distribution were investigated, and five were chosen, including versions of three older test statistics, to wit, Lilliefors-Kolmogorov-Smirnov (Lilliefors, 1969), Cramer-von Mises with K transformation (Seshadri, Csorgo, and Stephens, 1969), and Shapiro-Wilk (Metz, Haccou, and Meelis, 1994) statistics, and two more contemporary test statistics, the Gini statistic from econometrics (Gail and Gastwirth, 1978) and the Kullbach-Liebler discrimination statistic from information theory with Correa's entropy estimator (Choi, Kim, and Song, 2004; and see Kullback, 1968 and additional references in Chapter 2, *supra*). The programming of each of these tests was accomplished and verified by the researcher, the results being found in close accord with critical value tables presented in the literature. For each of these tests, critical values were determined anew for this study, in a similar fashion to the procedure for MADMASR. The results were tabulated in Table 5.

Robustness of the competing tests.

In order to test the robustness of the competing tests with respect to Type I error rate, critical values for selected sample sizes and alpha levels, as shown on page 1 of Table 5, would be saved and new samples from the exponential distribution simulated with parameters equal to, respectively, 0.1, 0.5, 2 and 10. Using the stored critical values, 100,000 simulated random samples would be taken for each case and the rejections at each upper critical level counted, resulting in a Monte Carlo estimate of the true Type I error rate for each procedure and case. This step was omitted, insofar as the competing tests have been well studied in the literature. Robustness with respect to Type II error may be treated in conjunction with discussion of the results observed from the power study.

Comparative power of the competing tests.

Sample sizes, n = 4, 10, 20, and 50, with critical values from Table 5, were utilized to investigate the power of the competing tests for exponentiality. The same 20 alternative distributions on which the MADMASR statistic was tested were used here for Monte Carlo sampling, with 100,000 trials for each distributional source and sample size, and for each upper critical value, resulting in a proportion of rejections for each case. These rejection rates, giving a Monte Carlo estimate of the true power for each procedure and case, are set out in Table 5.

Mathematical Details and Specifications

The Theta-Star Estimate: RANGEN, the IMSL FORTRAN 90 MP Library, and Mixed Distributions

The mathematical development, formulae, and algorithm (summarized above) for estimating the exponential parameter for a pseudo-random sample (generated with known parameter) is presented in an appendix on the Theta-Star estimator. There is considerable literature, treated in Chapter 2, investigating the rejection rates of various tests for exponentiality when applied to data resulting from simulated random sampling from non-exponential distributions. The RANGEN (Blair, 1987) module furnishes FORTRAN subroutines to simulate random samples from the Cauchy, Chi-Square, Erlang, exponential, F, Laplace, lognormal, standard normal, t, and unit-uniform distributions, not all of which are utilized in this study or, typically, in goodness-of-fit studies for exponentiality. The IMSL FORTRAN Library provides routines for computing the gamma function and for simulated random sampling from several distributions, including the gamma and Weibull distributions among them. The Dataplot system (Filliben & Heckert, 1978), which is available from the National Institute of Standards and Technology, is used here, and will be treated, in connection with the Makeham distribution.

Sampling from mixed distributions can be simulated by selecting a percentage of the sample from one distribution and the complementary percentage from a second distribution, which was done for the first-mentioned alternatives above (Gentle, 2003, p. 110; Meeker & Escobar, 1998, 116-117; and see Lawless, 2003; Lindsay, 1995). The sample may be combined from two parts or a single uniform sample may be obtained, with alternate reference to inverse cumulative distribution functions for the two desired distributions (if a balanced mixture is sought). In lieu of mixing definite proportions, a probabilistic switch between alternative components can be used, as will be discussed in connection with the first, second, and twentieth alternative distributions.

The Alternative Distributions Replicated For Each Sample

The Mixture of two exponential distributions.

The first alternative is a mixture, half from the exponential distribution with mean equal to 0.5 and half from that with mean equal to 4.0. Figure 2 displays a simulation of the first alternative distribution with 20,000 variates. The unit uniform distribution is used as a probability switch. A single random variate is drawn. If the variate is less than 0.5, a variate for the mixture distribution is drawn from the exponential with mean equal to 0.5, otherwise a variate for the mixture is drawn from the exponential with mean equal to 4.0. This process is repeated for the desired sample size. RANGEN (Blair, 1987) is the random generator for obtaining both uniform and exponential variates.

The mean of the mixture distribution is easily found, as E[X] =

$$\int_0^\infty x\{.5 f_1(x) + .5 f_2(x)\}dx = .5\int_0^\infty x f_1(x)\,dx + .5\int_0^\infty x f_2(x)\,dx = .5\{\mu_1 + \mu_2\} = .5\{.5 + 4\} = 2.25,$$

for the mixture here. The variance is found as E[(X - 2.25)^2] =

$$\int_0^\infty (x - 2.25)^2\{.5 f_1(x) + .5 f_2(x)\}dx = .5\int_0^\infty \left(x^2 - 4.5x + 5.0625\right)\left\{\left(\frac{e^{-x/0.5}}{0.5}\right) + \left(\frac{e^{-x/4}}{4}\right)\right\}dx =$$

11.1875. As may be seen from Table 4, the two average moments for 100,000 simulated random samples at each of four sample sizes are consistent with the theoretical values.

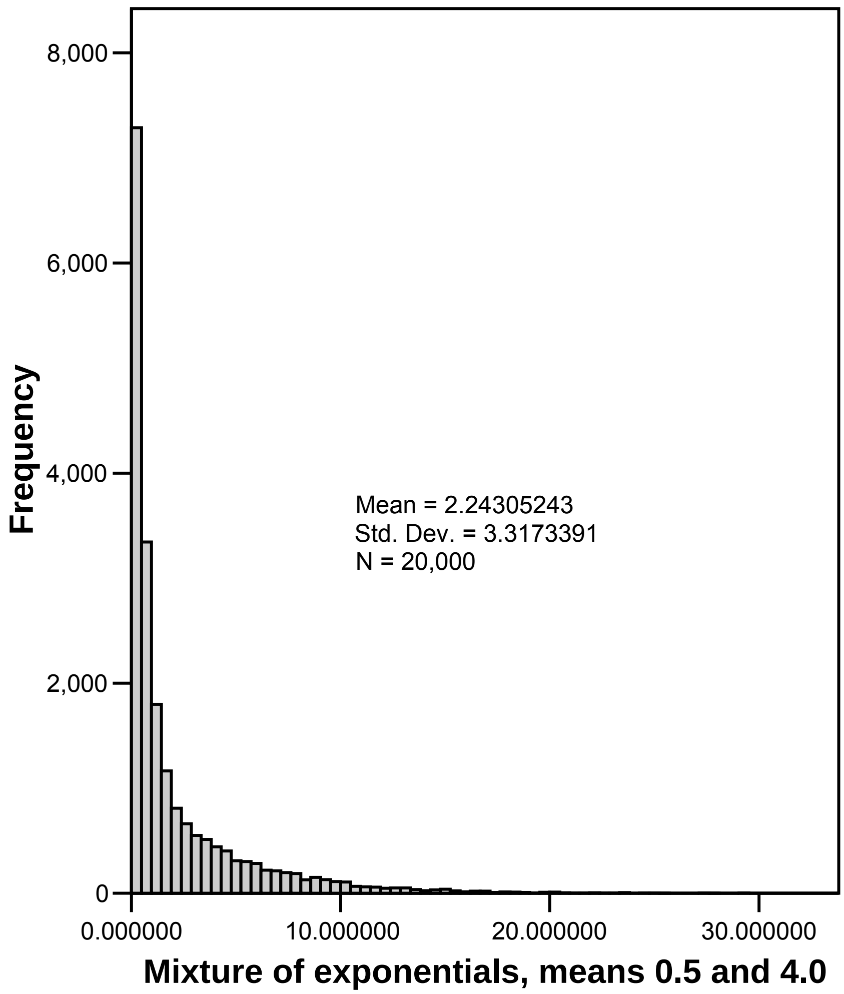

Figure 2. Histogram of a simulation of 20,000 variates from alternative distribution no. 1 (simulation in MINITAB; graph in SPSS.) The simulation was achieved by combining 10,000 variates each from the exponential distributions with means 0.5 and 4, respectively.

The Mixture of exponential and normal distributions.

The second alternative is a mixture half from the exponential distribution with mean of unity and half from N(10, 1). Figure 3 displays a simulation of the second alternative distribution with 20,000 variates. The unit uniform distribution is used as a probability switch. A single random variate is drawn. If the variate is less than 0.5, a variate for the mixture distribution is drawn from the exponential distribution. Otherwise a variate for the mixture is drawn from the Normal distribution. This process is repeated for the desired sample size. RANGEN (Blair, 1987) is the random generator for obtaining the uniform, exponential, and standard normal variates. The latter are transformed for the parameters desired.

The mean of the mixture distribution is E[X] = $\int_{-\infty}^{\infty} x\left\{.5 f_1(x) + .5 f_2(x)\right\} dx$ = $.5\int_0^{\infty} x f_1(x)\,dx + .5\int_{-\infty}^{\infty} x f_2(x)\,dx = .5\left\{\mu_1 + \mu_2\right\} = .5\left\{1+10\right\} = 5.5$ for this mixture. The variance is E[(X – 5.5)^2] = $.5\int_{-\infty}^{\infty}\left(x^2 - 11x + 30.25\right)\left(\exp(-x) + \frac{\exp\left(-\frac{(x-10)^2}{2}\right)}{\sqrt{2\pi}}\right)dx$ =

$$.5\int_0^{\infty}\left(x^2 - 11x + 30.25\right)\left(e^{-x}\right)dx \; + .5\int_{-\infty}^{\infty}\left(x^2 - 20x + 100\right)\left(\frac{\exp\left(-\frac{(x-10)^2}{2}\right)}{\sqrt{2\pi}}\right)dx \; + \; .5$$

$$+ .5\int_{-\infty}^{\infty}(9x)\left(\frac{\exp\left(-\frac{(x-10)^2}{2}\right)}{\sqrt{2\pi}}\right)dx \; + \; .5\int_{-\infty}^{\infty}(-69.75)\left(\frac{\exp\left(-\frac{(x-10)^2}{2}\right)}{\sqrt{2\pi}}\right)dx$$ Using

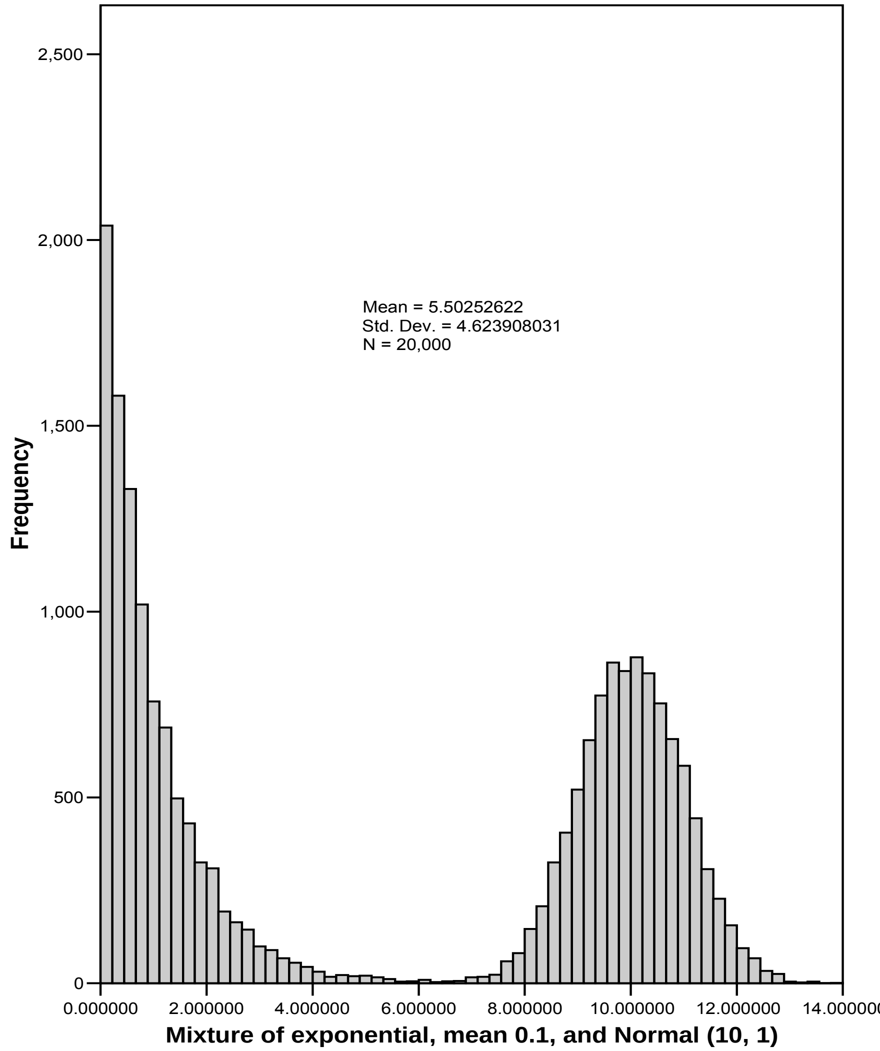

Figure 3. Histogram of a simulation of 20,000 variates from alternative distribution no. 2 (simulation in MINITAB; graph in SPSS.) The simulation was achieved by combining 10,000 variates each from the exponential distribution with mean of unity and the normal distribution with mean of 10 and standard deviation of unity.

integration by parts for the first integral yields $.5\left(2-11+30.25\right)=10.625$, while the remaining three integrals yield $.5\left(Var\left[X_2\right]+9E\left[X_2\right]-69.75\left[1\right]\right)=.5\left(1+90-69.75\right)=$ 10.625, where $X_2 \sim N\left(10,1\right)$, so that the variance of the mixture distribution equals 21.25. As may be seen from Table 4, the two average moments for 100,000 simulated random samples at each of four sample sizes are consistent with the theoretical mean and variance.

The gamma distribution (third and fourth alternatives).

The gamma distribution is useful in engineering applications due to its flexibility of shape as its parameters change value. The distribution is given by the pdf, $f\left(x;\sigma,\lambda\right)=\frac{1}{\sigma\Gamma\left(\lambda\right)}\left(\frac{x}{\sigma}\right)^{\lambda-1}\exp\left[-\left(\frac{x}{\sigma}\right)\right]$, for $x \geq 0;\ \sigma,\lambda > 0$, where λ and σ are the shape and scale parameters respectively, and where $\Gamma\left(\cdot\right)$ is the gamma function, easily evaluated as $\Gamma\left(a\right)=\left(a-1\right)!$ for positive integers and otherwise using computer programs including MINITAB. The cdf cannot be expressed in closed form. The mean is given by the product of its parameters, $\sigma\cdot\lambda$, so that it is a simple matter to assure an approximate value of unity. The variance is given by $\sigma^2\cdot\lambda$ (Bury, 1999, pp.208-210).

The gamma distribution reduces to a chi-square distribution with degrees of freedom equal to 2λ, when the scale parameter is set equal to two. When the shape parameter is a positive integer, the gamma distribution becomes the Erlang distribution, used to model the waiting time to the λth Poisson event. When the shape parameter equals unity, the gamma distribution reduces to the exponential distribution, with parameter σ (Bury, 1999, pp. 213-214).

Simulating a gamma variate, x, is easy if λ is an integer, in which case a gamma variable is the sum of λ exponential variables with parameter σ. In this case, an inverse cdf can be applied to a vector of λ random variates, u_i, from the unit uniform distribution: let $x = -\sigma \sum_{i=1}^{\lambda} \ln(u_i)$ (Bury, 1999, p. 211). Because RANGEN (Blair, 1987) provides for pseudo-random sampling from the Erlang distribution, the relevant subroutine in this FORTRAN module was employed to obtain samples from gamma with integer value of λ, for this investigation.

For non-integer shape parameter, as in the first simulated distribution used herein, several techniques for simulation have been presented, most being variations of the acceptance-rejection method. Different approaches are optimal for specific ranges of the shape parameter. See Rubinstein (1981, pp. 71-80), Bury (1999, pp. 211-212), Gentle (2003, pp. 178-183), and see Press et al. (1992, pp. 282-283). Given the large number of samples to be simulated, the researcher selected the concise Cheng-Feast algorithm, available for generating gamma variates when λ > 1, as implemented by Gentle (Cheng & Feast, 1979; Gentle, 2003, p. 179). The algorithm entails taking two random variates from the unit uniform distribution and then performing two logical tests in succession, utilizing the shape parameter of the desired gamma distribution. This process is repeated in the case neither test succeeds. In the case of acceptance under either test, the resulting variate has a ‘reduced’ gamma distribution with scale variable of unity. The desired gamma distribution is achieved simply, multiplying the generated variate by the desired value of σ (Bury, 1999, p. 209). The algorithm is incorporated in the author’s Monte Carlo FORTRAN program, mcexp_rejectrates. The FORTRAN module RANGEN (Blair, 1987) was used to obtain uniform variates.

Two different gamma densities were selected, $f\left(x;\, \sigma = 0.625,\, \lambda = 1.6\right)$ and $f\left(x;\, \sigma = 0.25,\, \lambda = 4.0\right)$, the reciprocal parameters resulting in expected values of unity for each. The respective variances are $\sigma^2 \cdot \lambda \;=\; \sigma \cdot 1 \;=\; 0.625$ and $\sigma^2 \cdot \lambda \;=\; \sigma \cdot 1 \;=\; 0.25$. The average statistics shown in Table 4 for all replications at each sample size for these two distributions are consistent with these theoretical values.

The Weibull distribution (fifth through seventh alternatives).

The Weibull distribution, useful in reliability engineering, is a two-parameter, continuous distribution with support on the positive x-axis. Unfortunately, the literature uses various notation schemes – some of which assign different meaning to the same letters – sometimes within the same work (e.g., Lawless, 2003), where different segments of the text may feature distinct attributes of the distribution. In order to emphasize the geometric attributes of scale and shape, we shall follow Bury (1999; pp. 112-113, 311-316), giving the pdf as $f\left(x;\sigma,\lambda\right) \;=\; \dfrac{\lambda}{\sigma}\left(\dfrac{x}{\sigma}\right)^{\lambda-1} \exp\left[-\left(\dfrac{x}{\sigma}\right)^{\lambda}\right]$, for $x \ge 0,\, \sigma > 0,\, \lambda > 0$, and the cdf as

$F(x;\sigma,\lambda) = 1 - \exp\left[-\left(\frac{x}{\sigma}\right)^{\lambda}\right]$, where σ is the scale parameter and λ is the shape parameter. With shape parameter equal to unity, the Weibull distribution reduces to the exponential distribution. (A similar reduction of the Weibull to the Rayleigh distribution holds. There is also a three parameter Weibull distribution, with which we shall not be concerned, obtained by subtracting a location parameter from each appearance of the variate, x, in the pdf.) The r^{th} moment about the origin of the Weibull distribution is given by $\mu_r' = \sigma^r\,\Gamma\left(1+\frac{r}{\lambda}\right)$, where $\Gamma(\cdot)$ is the gamma function. The mode of the Weibull distribution equals $\sigma\left(\frac{\lambda-1}{\lambda}\right)^{\frac{1}{\lambda}}$ and the quantile of order q equals $x_q = \sigma\left[\ln\left(\frac{1}{1-q}\right)\right]^{\frac{1}{\lambda}}$. Employing the quantile, a random sample from the Weibull distribution can be simulated, as was done in the present study, using the inverse cdf method after simulating a random sample from the uniform distribution on the unit interval, with variate μ_i, in which case the Weibull variate may be obtained as $x_i = \sigma\left[\ln\left(\frac{1}{\mu_i}\right)\right]^{\frac{1}{\lambda}}$ (Bury, 1999, pp. 311-317). Skewness and kurtosis for a Weibull distribution are given as rational functionals of gamma functions of the shape parameter (Bury, 1999, p. 314). RANGEN (Blair, 1987) was used for the required uniform variates.

The expected value of a Weibull distribution is given by $\sigma\cdot\Gamma\left(1+\frac{1}{\lambda}\right)$, so that on selecting values for the shape parameter, λ, an expected value of unity can be assured[26] by taking $\sigma = \dfrac{1}{\Gamma\left(1+\frac{1}{\lambda}\right)}$. The variance is given by $\sigma^2\left[\Gamma\left(1+\dfrac{2}{\lambda}\right) - \Gamma^2\left(1+\dfrac{1}{\lambda}\right)\right]$ (Bury, 1999, p. 313). MINITAB was used to evaluate the gamma function, in this process. The three Weibull distributions selected for treatment may be described as follows, in Table I.

Table I

Three Weibull Alternative Distributions

λ [shape]	σ [scale]	E[X]	Var.[X]
0.5	$\sigma = \dfrac{1}{\Gamma\left(1+\frac{1}{.5}\right)} = 0.5$	1.0	$(.5)^2\left[\Gamma\left(1+\dfrac{2}{.5}\right) - \Gamma^2\left(1+\dfrac{1}{.5}\right)\right] = 5$
1.5	$\sigma = \dfrac{1}{\Gamma\left(1+\frac{1}{1.5}\right)} = 1.10773$	1.0	$(1.10773)^2\left[\Gamma\left(1+\dfrac{2}{1.5}\right) - \Gamma^2\left(1+\dfrac{1}{1.5}\right)\right]$ $= 0.46100$
2.0	$\sigma = \dfrac{1}{\Gamma\left(1+\frac{1}{2}\right)} = 1.12838$	1.0	$(1.12838)^2\left[\Gamma\left(1+\dfrac{2}{2}\right) - \Gamma^2\left(1+\dfrac{1}{2}\right)\right]$ $= 0.27324$

The average means and variances for all replications, in all cases of the Weibull

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[26] Approximately, for the purposes of computer simulation.

distribution shown in Table 4, are consistent with these theoretical values.

The Gumbel distribution (eighth alternative).

This is the only alternative distribution used herein with support on the negative axis. Bury notes that, in the typical engineering application, the empirical distribution is shaped so that negative values of the variable are negligible. Whereas the Weibull distribution arises as a limiting distribution for the minimum of a random sample, the Gumbel distribution[27] is a limiting distribution for the sample maximum, both thus belonging to the genre of extreme value distributions. Bury gives the pdf of a Weibull distribution with location parameter, μ, and scale parameter, σ, as

$f\left(x;\mu,\sigma\right) = \frac{1}{\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right) - \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}$, for $-\infty < x,\ \sigma > 0,\ \mu < \infty$, and the cdf as $F\left(x;\mu,\sigma\right) = \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}$. A variate, X, with Gumbel distribution is related to a Weibull variate, Y, by the relation $X_{\mu,\sigma} = \ln\left[\frac{1}{Y_{\sigma_W,\lambda_W}}\right]$, with $\mu = \ln\left[\frac{1}{\sigma_W}\right]$ and $\sigma = \frac{1}{\lambda_W}$. The moments are obtained from gamma functions. The mode of a Gumbel distribution equals the location parameter, μ, and the

[27] MINITAB refers to the Gumbel as the “largest extreme value” distribution.

quantile of order q is given by $x_q = \mu - \sigma\left\{\ln\left[\ln\left(\frac{1}{q}\right)\right]\right\}$. The foregoing attribute leads to the method used here for simulating a random sample from the Gumbel distribution by the inverse cdf transformation, yielding $x_i = \mu - \sigma \ln\left[\ln\left(\frac{1}{\mu_i}\right)\right]$, where μ_i is a uniform random variate on (0, 1) (Bury, 1999, pp. 267-271). RANGEN (Blair, 1987) was used to obtain unit uniform variates.

This distribution resulted in underflow in the subroutine calculating the MADMASR test statistic, when values expected to be positive occurred extremely close to zero. Use of the 'TINY' function in FORTRAN, producing the smallest positive real number registered by the computer, in that subroutine appeared to reasonably, but not always, solve this problem.[28]

A single Gumbel distribution was simulated, that with location parameter, μ, equal to 0.42278 and scale parameter, σ, equal to unity. The expected value of a Gumbel distribution is given by $\mu + \gamma\sigma$, where $\gamma \approx 0.57722$ is Euler's constant. The variance is given by $\frac{\pi^2\sigma^2}{6} \approx 1.64493\sigma^2$ (Bury, 1999, pp.269-270). Thus, the theoretical moments for the Gumbel alternative here are

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[28] Rigorous study of the conditions of underflow, of any trade-offs between the elimination thereof and loss of precision in the procedures used, and of more reliable methods for avoiding underflow lies beyond the scope of this investigation.

$E[X] \approx 0.42278 + 0.57722(1.0) = 1.0$ and $Var[X] = \dfrac{\pi^2 \cdot 1^2}{6} \approx 1.64493$, with which the average statistics for 100,000 replications shown in Table 4 are consistent.

The lognormal distribution (ninth through eleventh alternatives).

If the natural logarithm of a variate is normally distributed, then the variate itself has a lognormal distribution. The parameters of the normally distributed logarithm with mean, μ, and standard deviation, σ, can be related to the pdf of the lognormal distribution, given as follows, with the notation for the normal parameters preserved:

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\} = \frac{1}{x\sigma}\phi\left(\frac{\ln(x)-\mu}{\sigma}\right), \text{ for}$$

$x > 0,\ \sigma > 0, -\infty < \mu < \infty$, and $\phi(\cdot)$ is the standard normal density function. However, the lognormal model differs in the character of its parameters: $\exp(\mu)$ is the scale parameter and σ is the shape parameter. The cdf is given by

$$F(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{t} \exp\left\{-\frac{1}{2}\left(\frac{\ln(t)-\mu}{\sigma}\right)^2\right\} dt = \frac{1}{x\sigma}\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right), \text{ where } \Phi(\cdot)$$

is the standard normal cdf (Bury, 1999, pp. 154-155; and see Lawless, 2003, pp. 21-23). The lognormal distribution can be used to model survival in different situations, including mixtures of long and short lifetimes as in survival after treatment for cancer and duration of marriage (Lawless, 2003, pp. 22-23).

A random sample from the lognormal distribution can be simulated by generating a random sample $\{z_i\}$ from the standard normal distribution and applying the transformation, $x_i = \exp(\mu + \sigma z_i)$ (Bury, 1999, pp. 157). However, as RANGEN (Blair, 1987), includes a subroutine (lnor1) which generates lognormal variates, this module was used for simulating the lognormal samples in this study.

The Expected value and variance of a lognormal distribution are given respectively by $\exp\left(\mu + \frac{\sigma^2}{2}\right)$ and $\exp(2\mu + \sigma^2)\cdot\left[\exp(\sigma^2) - 1\right]$ (Bury, 1999, pp. 155-156). The three lognormal alternative distributions in the current study may be described as in Table II below.

Table II

Three Lognormal Alternative Distributions

μ	e^{μ} (scale)	σ (shape)	mean	variance
0.0	1.0	0.2	$e^{0.02} \approx 1.02020$	$e^{0.04}\left(e^{0.04} - 1\right) \approx 0.04248$
0.0	1.0	1.0	$e^{0.5} \approx 1.64872$	$e(e-1) \approx 4.67077$
0.0	1.0	2.0	$e^{2} \approx 7.38906$	$e^{4}\left(e^{4} - 1\right) \approx 2926.35984$

The average statistics for 100,000 replications in each case, shown in Table 4, are consistent with these theoretical values for the sampled distributions.

The Makeham distribution (twelfth alternative).

This distribution is sometimes referred to as the Gompertz-Makeham distribution. However, the mortality law of Gompertz from actuarial science is a special case of Makeham's mortality law, in which one parameter vanishes, reducing the distribution to two parameters (Scollnik, 1995, pp. 410-411; see, also, Barlow & Proschan, 1981; Chiang, 1984, who derives the Gompertz and Makeham distributions in the context of mortality; and Gnedenko, 1966). The law of human mortality published by Benjamin Gompertz in 1825 for use in actuarial life tables allowed for only deterioration, of the two causes for death hypothesized, the other cause being chance. Gompertz' law "assumed that an individual's resistance to death decreases at a rate proportional to the force of resistance itself" (Chiang, 1984, p. 198). In 1869, Makeham modified the law of mortality suggested by Gompertz, "restoring the missing 'chance' component" (Chiang, 1984, p. 199). Although Makeham's law is both ubiquitously acknowledged and critiqued in the science of demography (Keyfitz, 1977; Keyfitz & Beekman, 1984; Preston, Heuveline, & Guillot, 2001; Schoen, 1988), the corresponding distribution is only sketchily treated in the literature as a probability density function.

Chiang gives the pdf and cdf of a Makeham distribution, respectively, as

$f(x) = \left[A + Bc^x\right]\exp\left\{-\left[Ax + b\left(c^x - 1\right)/\ln c\right]\right\}$ and

$F(x) = 1 - \exp\left\{-\left[Ax + b\left(c^x - 1\right)/\ln c\right]\right\}$ (Chiang, 1984, p. 199). If the substitution, $c = e^{\gamma}$ is made, then the pdf becomes

$$f(x) = \left[A + Be^{\gamma x}\right]\exp\left\{-\left[Ax + \frac{B}{\gamma}\left(e^{\gamma x} - 1\right)\right]\right\}$$

and the cdf becomes $F(x) = 1 - \exp\left\{-\left[Ax + \frac{B}{\gamma}\left(e^{\gamma x} - 1\right)\right]\right\}$.

Scollnik (1995) treated methods for simulating a random variate from the Makeham distribution, including the rejection sampling method. Alternatively, a solution can be found for the inverse of the Makeham cdf, in terms of the Omega, or Lambert's W function. Approximation of the W function has been investigated for real values (Barry, Culligan-Hensley, & Barry, 1995; Barry, Barry, & Culligan-Hensley, 1995). The principal branch of the W function can be expressed as the power series, $W_0(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!}$ (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996, p. 339).

More detailed analysis of the Makeham distribution was provided by Meeker and Escobar (1998, pp. 108-109) and through computer functionality using the DATAPLOT system, by Filliben (1978). The Makeham parameterization resulting after the $c = e^{\gamma}$ substitution, above, is reflected in the first parameterization furnished by Meeker and Escobar (for which authors, Filliben named the default parameterization used in DATAPLOT), yielding the cdf

$F(x;\, \gamma_{\text{Meeker}}, \kappa_{\text{Meeker}}, \lambda_{\text{Meeker}}) = 1 - \exp\left[-\frac{\lambda\kappa x + \gamma\exp(\kappa x) - \gamma}{\kappa}\right]$, for

$x > 0;\ \gamma,\ \kappa > 0;\ \lambda \geq 0$, in which the parameters have dimension which is the reciprocal of that of x (Meeker & Escobar, 1998, p. 108; and see online documentation for at http://www.itl.nist.gov/div898/software/dataplot/, as accessed on July 26, 2005). A second parameterization given by Filliben and attributed to the Digital Library of Mathematical Functions (DLMF), which gives pdf and cdf respectively for the Makeham distribution respectively as

$f\left(x;\ \xi_{DLMF}, \lambda_{DLMF}, \theta_{DLMF}\right) = \xi\lambda\left(\theta + e^{\lambda x}\right)\exp\left(-\xi\left(e^{\lambda x} - 1\right) - \xi\theta\lambda x\right)$ and

$F\left(x;\ \xi_{DLMF}, \lambda_{DLMF}, \theta_{DLMF}\right) = 1 - \exp\left(-\xi\left(e^{\lambda x} - 1\right) - \xi\theta\lambda x\right)$, for

$x > 0;\ \lambda,\ \xi,\ \theta > 0$ (Filliben, 1978), was used. It can be seen that the following equivalences hold, with ‘m’ referring to the Meeker-Escobar parameterization, ‘D’ referring to the DLMF parameterization, and with γ_{Chiang} in the substitution into the ABc parameterization above being equivalent to κ_m: $A = \lambda_m = \theta_D\lambda_D\xi_D$; $c = e^{\kappa_m} = e^{\lambda_D}$; and $B = \gamma_m = \lambda_D\xi_D$.

Meeker and Escobar proceeded to another parameterization separating a scale parameter and two dimensionless shape parameters. Some graphical analysis, including the hazard function, is given (1998, pp.108-109). None of the sources gave closed form for the moments of the Makeham distribution. However, for the single instance of this distribution used in this study, setting $\theta_D = \lambda_D = \xi_D = 1$, the mathematical program, Maple evaluated these numerically, resulting in expected value of approximately 0.40365 and variance of approximately 0.11244, which are consistent with the average statistics for

100,000 replications for each sample size, shown in Table 4. A simulation of the Makeham distribution with DLMF parameters each equal to unity is displayed in Figure 4.

The most efficient way to simulate random samples from the Makeham distribution was to utilize the random number facility for this distribution in the DATAPLOT system[29]. This entailed running a sequence of DATAPLOT commands[30] incorporating a loop for 100,000 replications and, for each selected sample size, storing the required number of variates in a sequential text file, to be read, one sample at a time, by the author's FORTRAN program, mcexp_rejectrates.

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[29] See footnote 25, above.

[30] The code, for example, for samples size 4 was as follows:

```
set gompertz makeham definition dlmf
let xi = 1.0
let lambda = 1.0
let theta = 1.0
let n = 4
set write rewind off
set write format 4f15.12
loop for k = 1 1 100000
let z = gompertz makeham random numbers for i = 1 1 n
write ten5thmakehamn4.txt z for i = 1 1 n
end of loop
```

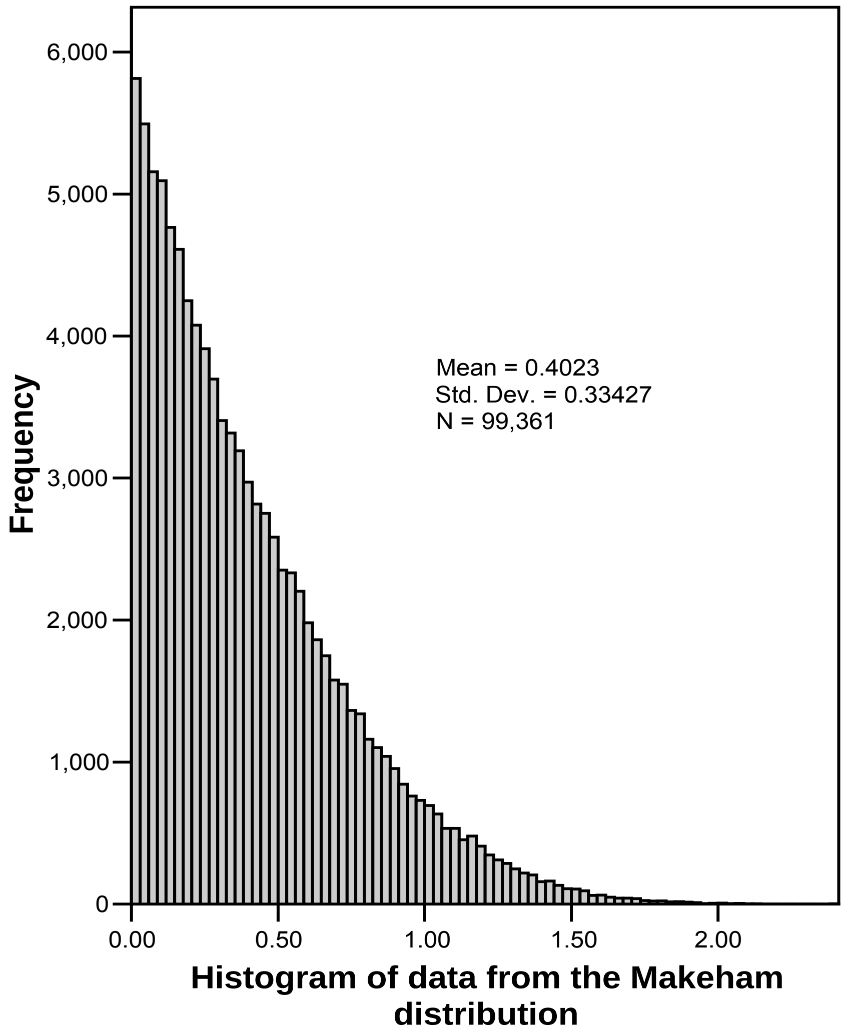

Figure 4. Histogram of alternative distribution no. 12, the Makeham distribution with all three parameters equal to unity. The variates were simulated in DATAPLOT. The graph was prepared in SPSS by sampling from the researcher's text file holding 400,000 simulated variates representing 100,000 Monte Carlo samples size n = 4.

The Chi-square distribution (thirteenth through sixteenth alternatives).

The Chi-square distribution family results as a special case of the gamma distribution, when the scale parameter, σ, is set as the constant, 2.0. A notational change, replacing the shape parameter, λ, by $\frac{\upsilon}{2}$, results in the pdf,

$f\left(x;\,\upsilon\right) \;=\; \dfrac{1}{2\Gamma\left(\frac{\upsilon}{2}\right)}\left(\dfrac{x}{2}\right)^{\frac{\upsilon}{2}-1}\exp\left(-\dfrac{x}{2}\right)$, for $x \geq 0,\; \upsilon > 0$, where the parameter, υ represents the number of degrees of freedom in the context of sampling distributions utilizing the Chi-square distribution (Bury, 1999, p. 213). For $\upsilon = 2$, the distribution reduces to the exponential distribution with mean of 2.0. Following from the attributes of the gamma distribution, the expected value and variance, respectively, of a Chi-square distribution are υ and 2υ. The FORTRAN module, RANGEN (Blair, 1987) was used to simulate samples from the Chi-square distribution.

Four instances of the Chi-square distribution were utilized in this study, described as follows in Table III.

Table III

Four Chi-square Alternative Distributions

υ	mean	variance	
1	1	2	
2	2	4	Note: this Chi-square distribution is equivalent to the one-parameter exponential distribution with scale parameter equal to 2.0. It is included to present an uninterrupted sequence of values of the parameter, υ
3	3	6	
4	4	8	

The average sample means and variances for 100,000 replications of each case are consistent with these theoretical values, as may be seen from Table 4.

The uniform distribution (seventeenth alternative).

The uniform distribution simulated here is sometimes called the unit uniform (continuous) distribution, with constant density and support restricted to the ‘unit’ interval, [0, 1]. Despite its simplicity, this distribution can be seen as a special case of the two-parameter beta distribution, with support on the open unit interval, (0, 1), which has considerable flexibility in appearance. The latter distribution has the pdf, $f\left(x;\ \lambda_1, \lambda_2\right) = \dfrac{\Gamma\left(\lambda_1+\lambda_2\right)}{\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)} x^{\lambda_1-1}\left(1-x\right)^{\lambda_2-1}$, for $0 < x < 1,\ \lambda_1, \lambda_2 > 0$. If $\lambda_1 = \lambda_2 = 1$, then the unit uniform distribution results, with pdf, $f\left(x\right) = 1$ and cdf, $F\left(x\right) = x$, for $0 \le x \le 1$. The expected value and variance are $\frac{1}{2}$ and $\frac{1}{12}$ respectively. Simulation was accomplished by the researcher, using the uni1 subroutine in the module, RANGEN (Blair, 1987). The average statistics for 100,000 simulated random samples for each sample size are consistent with the theoretical values for the unit uniform distribution, as may be seen in Table 4.

The half-normal distribution (eighteenth alternative).

This distribution, sometimes called the folded-over normal distribution, can be understood from the perspective of random sampling, or in the Monte Carlo process, of simulating random variates. For present purposes, the underlying normal distribution used was the standard normal distribution. Sampling was accomplished by, first, using the RANGEN subroutine, normb1, to obtain a normally distributed variate and then taking the absolute value (See Gentle, 2003). The pdf of the half-normal distribution derived from N(0, 1) is given by

$f\left(x;\ \mu=0,\ \sigma=1\right) \ = \ 2\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\right\} \ = \ \sqrt{\frac{2}{\pi}}\ \exp\left(-\frac{x^2}{2}\right)$, for $x \geq 0$. The expected value is obtained by integration:

$$E\left[X\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} x\ \exp\left(-\frac{x^2}{2}\right)dx \ = \ \sqrt{\frac{2}{\pi}}\ \left[-\exp\left(-\frac{x^2}{2}\right)\right]_0^{\infty} \ = \ \sqrt{\frac{2}{\pi}} \ \approx \ 0.79788 .$$

The variance can be obtained as $E\left[X^2\right]-\left(E\left[X\right]\right)^2$, where $E\left[X^2\right]$ can be evaluated, using integration by parts, as equal to unity. Then the variance

$= 1 - \left(\sqrt{\frac{2}{\pi}}\right)^2 \ = \ 1 - \frac{2}{\pi} \ \approx \ 0.36338$. The average sample mean and average sample variance for 100,000 replications for each case are consistent with these theoretical values for the half-normal distribution, as may be seen in Table 4.

The Anomalous Alternative Distributions Represented by Finite Populations

The process of constructing the populations.

In order to represent the anomalies of real-world data beyond the variety of alternative distributions 1 through 18, two finite populations, each of size N = 1,500, were simulated using two different mixture methods for the same two distributional components, Beta (1.1, 0.4), the variate for which will be designated by ‘B’, and the triangular distribution with support on the closed, real interval [0.2, 0.7] and mode at 0.6, the variate for which will be designated by ‘T’. In addition, a Bernoulli variate, with probability equal to 0.5, will be designated by ‘b’.

These two mixture distributions were constructed by simulating the component distributions and also combining the variates in MINITAB. Graphical facilities in MINITAB were used to explore combinations with suitable attributes, principal among them, anomalous appearance in shape. MINITAB and Excel were used to analyze the distributions. The respective populations of data were formatted in Excel as six place decimals, consistent with a FORTRAN format specification of F8.6. In Excel, the populations were then stored as tab-delimited text files, to be repetitively accessed for Monte Carlo sampling in the researcher’s

FORTRAN programs, testing for Type I error rate and power with results to be reported, respectively, in Tables 4 and 5.

In order to present some variety of distributional shape, two mixture methods were used. For the first population, a new variate was constructed as the sum of half of a Beta variate and half of a Triangular variate, signified by $X = 0.5B + 0.5T$. For the second population, a new variate was constructed as the sum of the product of a Bernoulli variate and a Beta variate and the product of the complement of the Bernoulli variate and a Triangular variate, signified by $X = bB + (1-b)T$. These populations will be treated and characterized statistically, after treatment of the Beta and Triangular components.

The Beta component distribution.

The Beta cdf cannot be expressed in closed form; its pdf is given as $f(x;\lambda_1,\lambda_2) = \dfrac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} x^{\lambda_1-1}(1-x)^{\lambda_2-1}$, for $0 < x < 1;\ \lambda_1,\lambda_2 > 0$. Both parameters are shape parameters, which stand in a symmetrical relationship to the pdf, to wit, $f(x;\lambda_1,\lambda_2) = f((1-x);\lambda_2,\lambda_1)$ (Bury, 1999, p. 238). The Beta distribution family is closely but indirectly related to the binomial distribution, due to the property that $\Gamma(n+1) = n!$ for integer values of n and reduces to the unit uniform distribution when $\lambda_1 = \lambda_2 = 1$. An advantage of the Beta distribution in generating random data is the great flexibility of shape which occurs for different combinations of parameters, depending upon whether one or both are less than or greater than unity. Symmetry is achieved, within the unit interval of support, if the parameters are equal, with kurtosis or peakedness varying with the size of the parameters. Asymmetry and skewness can be explored, with differing ratios of the respective parameters. If both parameters are less than unity, a 'U'-shaped distribution results. See Bury (1999, pp. 240-242).

The mean and variance of a Beta variable are given by $\mu = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$ and

$\sigma^2 = \dfrac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_2 + 1)}$ (Bury, 1999, p. 239; Miller & Miller, 1999, p. 211).

The Beta distribution selected by the researcher as a mixture component has parameters 1.1 and 0.4 and thus $\mu_B = \dfrac{1.1}{1.1 + 0.4} = 0.7\overline{3}$ and

$\sigma_B^2 = \dfrac{(1.1)(0.4)}{(1.1 + 0.4)^2 (1.1 + 0.4 + 1)} = 0.078\overline{2}$. The same pseudo-random selection of 1,500 Beta values was used for both anomalous distributions. The empirical mean and variance of this simulation of the Beta (1.1, 0.4) variate were $\overline{x}_B = 0.73897$ and $s_B^2 = 0.07683$.

The Triangular component distribution.

While a Triangular distribution can be selected with support on any real interval, for the researcher's purpose of mixture with a Beta distribution, an interval inside the unit interval, (0, 1) was used. If the closed interval of support is taken as [a, b], with mode at real value m inside the latter interval, then the cdf is given as $F(x;\, a,\, m,\, b) \;=\; \begin{cases} \dfrac{(x-a)^2}{(b-a)(m-a)},\; for\; a \le x \le m \\ 1 \;-\; \dfrac{(b-x)^2}{(b-a)(b-m)},\;\; for\; m \le x \le b \end{cases}$ (Evans, Hastings, & Peacock, 1993, p. 149). The distributional shape is a triangle with base coincident with the interval [a, b] and opposite vertex above x = m, where $a < m < b$ at a height equal to f(m), the density at x = m.[31] Thus, a pseudo-random sample from a particular Triangular distribution may be used to alter the shape of a pseudo-random sample from a particular Beta distribution, so as to achieve a mixture which does not appear to have any elegant mathematical representation but, rather, appears to be a collection of 'messy' empirical data.

The mean and variance of a Triangular distribution are given by $\mu \;=\; \dfrac{a\,+\,b\,+\,m}{3}$ and $\sigma^2 \;=\; \dfrac{a^2\,+\,b^2\,+\,m^2\,-\,ab\,-\,am\,-\,bm}{18}$ (Evans, Hastings, & Peacock, 1993, p. 149). The Triangular distribution chosen by the researcher has parameters 0.2, 0.7, and (mode) 0.6, so that the mean and variance are $\mu_T \;=\; \dfrac{0.2\,+\,0.7\,+\,0.6}{3} \;=\; 0.5$ and

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[31] Elishakoff (1999, p. 65) provides a centralized formulation of the triangular pdf: $f(x) \;=\; \begin{cases} A\left(1 \;-\; \dfrac{|x|}{a}\right),\;\; -a \,<\, x \,<\, a \\ 0,\;\; otherwise \end{cases}$, wherein $A = \dfrac{1}{a}$.

$$\sigma_T^2 = \frac{0.2^2 + 0.7^2 + 0.6^2 - (0.2)(0.7) - (0.2)(0.6) - (0.7)(0.6)}{18} = 0.009\overline{4}$$

respectively. The same pseudo-random selection of 1,500 Triangular values was used for both anomalous distributions. The empirical mean and variance of this simulation of the Triangular (0.2, 0.7, 0.6) variate were $\overline{x}_T = 0.49955$ and $s_T^2 = 0.01243$.

First anomalous mixture distribution (nineteenth alternative).

The first mixture distribution is constructed by combining two independently simulated variates from the Beta and Triangular distributions respectively. The two variates are taken in parallel sequence, according to their respective positions among the 1500 replications of each variate. The combination is a balanced, linear one, with $X_i = 0.5B_i + 0.5T_i$, for i = 1, ... , 1500. A histogram of this finite population is shown as Figure 5.

Statistical characterization of the variate, X, is uncomplicated. $E[X] =$

$E[0.5B + 0.5T] = 0.5E[B] + 0.5E[T] = 0.5(\mu_B + \mu_T) = 0.5(0.7\overline{3} + 0.5) = 0.61\overline{6}$.

From analysis in MINITAB of the components of this variate, as simulated, using notation for sample statistics to distinguish the theoretical value above, $\overline{x} =$

$0.5\overline{x}_B + 0.5\overline{x}_T = 0.5(0.73897 + 0.49955) = 0.61926$. In the researcher's programmed Monte Carlo procedure for testing this distribution, the average sample mean for 100,000 replications and for each sample size, n = 4, 10, 20, and 50, as shown in Table 4, is 0.619, consistent with the theoretical values, as

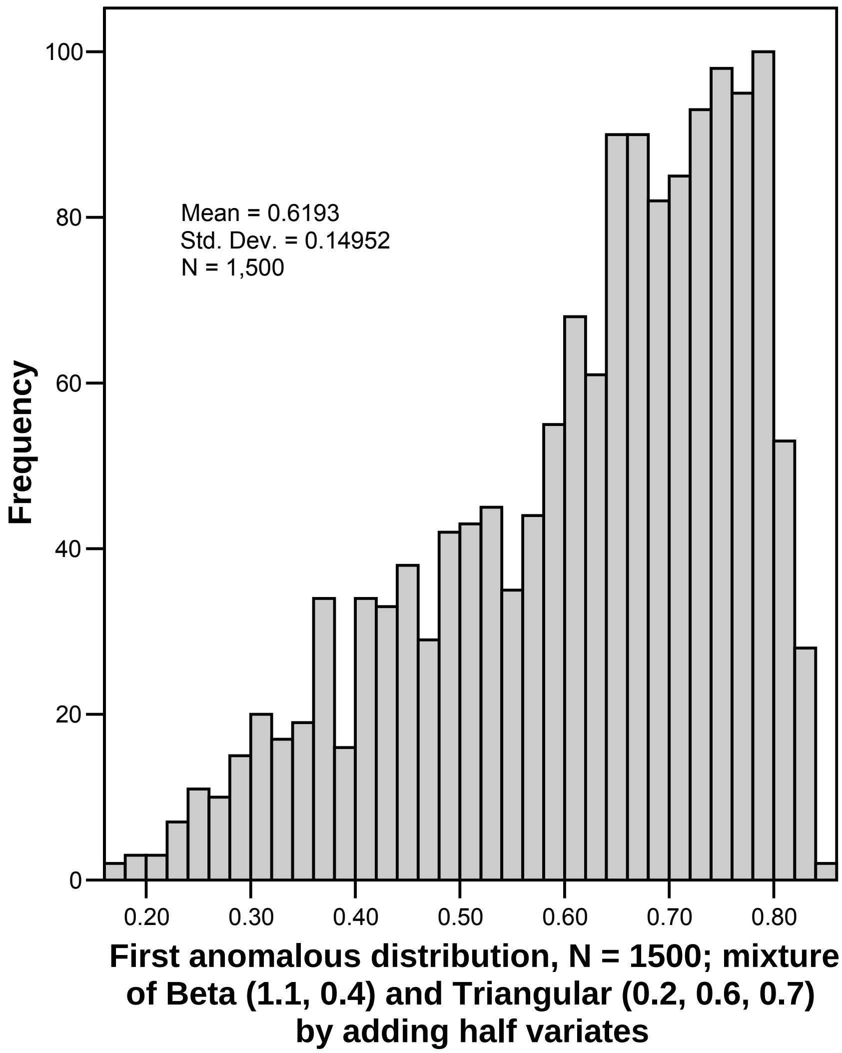

Figure 5. Histogram of alternative distribution no. 19 (a finite population of 1500 variates, as simulated in MINITAB; graph in SPSS.) The mixture was achieved by adding the product of 0.5 and a variate from the Beta (1.1, 0.4) distribution to the product of 0.5 and a variate from the Triangular (0.2, 0.6, 0.7) distribution.

derived both for the fundamental mixture distribution and for the component samples.

Because the variables B and T are stochastically independent[32],

$VAR\left[X\right] \;=\; VAR\left[0.5\,B + 0.5\,T\right] \;=\; 0.25\,VAR\left[B\right] \;+\; 0.25\,VAR\left[T\right]$

$= \; 0.25\left(0.078\overline{2} \; + \; 0.009\overline{4}\right) = \; 0.02191\overline{6}$. From analysis of the components in MINITAB, $s_x^2 \; = 0.25 s_B^2 \, + 0.25 s_T^2 \; = \; 0.25 \left(0.07683 + 0.01243\right) \; = \; 0.022315$. In the Monte Carlo procedure, the average sample variance for 100,000 replications and for each sample size, n = 4, 10, 20, and 50, as shown in Table 4, is 0.022, consistent with the theoretical values derived for the fundamental mixture distribution and for the component samples.

Second anomalous mixture distribution (twentieth alternative).

Statistical characterization of the second anomalous mixture is more complicated, due to stochastic dependence of the two product distributions involved. A third simulated variate, the Bernoulli with probability = 0.5, enters into this mixture. The new variate, X, may be represented as $X \; = \; b\,B + \left(1 - b\right)T$, where ‘b’ indicates the Bernoulli variate, which is simulated in 1500 replications, parallel to the Beta and Triangular variates. The latter comprise the same 1500

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[32] Notwithstanding any nonzero covariance between the samples generated and taken as ‘population’ components. In this simulation, MINITAB was used to find a covariance of 0.0000842 between the Beta and Triangular variates, a negligible amount resulting from sampling error.

replications combined in the first anomalous mixture distribution. A histogram of this finite population is shown as Figure 6.

Mixture distributions allow considerable variety and are generally a topic for specialized treatment (see Lindsay, 1995). Moreover, the two component product distributions must be investigated explicitly upon mathematical principles (see Galambos & Simonelli, 2004, and Heathcote, 2000, p. 114-115). Fortunately, the distribution under consideration is conducive to a fairly routine treatment, starting with derivation of the moments desired.

To discover the expected value of $X = bB + (1-b)T$, the components can be separated. For ease of notation in the remaining analysis of the distribution of X, integrals will be written without limits, with the understanding that integration is taken on the interval, $(0, \infty)$.

$E\left[bB\right] = \sum_{b=1,0} b\left(\int B \cdot f_B(B)\,dB\right) f_b(b) = 1(\mu_B)(0.5) + 0(\mu_B)(0.5) = 0.5\mu_B$, while

$E\left[(1-b)T\right] = \sum_{b=1,0} (1-b)\left(\int T \cdot f_T(T)\,dT\right) f_{(1-b)}(1-b) = 0(\mu_T)(0.5) + 1(\mu_T)(0.5) = 0.5\mu_T$.

Thus, $E\left[bB + (1-b)T\right] = E\left[bB\right] + E\left[(1-b)T\right] = 0.5\mu_B + 0.5\mu_T$. Applying this specifically to Beta(1.1, 0.4) and Triangular(0.2, 0.6, 0.7), $0.5\mu_B + 0.5\mu_T =$ $0.5\left(0.7\overline{3}\right) + 0.5(0.5) = 0.3\overline{6} + 0.25 = 0.61\overline{6}$. Additional insight is gained by taking an alternative course for the expectations of the respective product distributions, $X_1 = b \cdot B$ and $X_2 = (1-b) \cdot T$. Each of these variates is the product of independent random variables, although the two products are stochastically

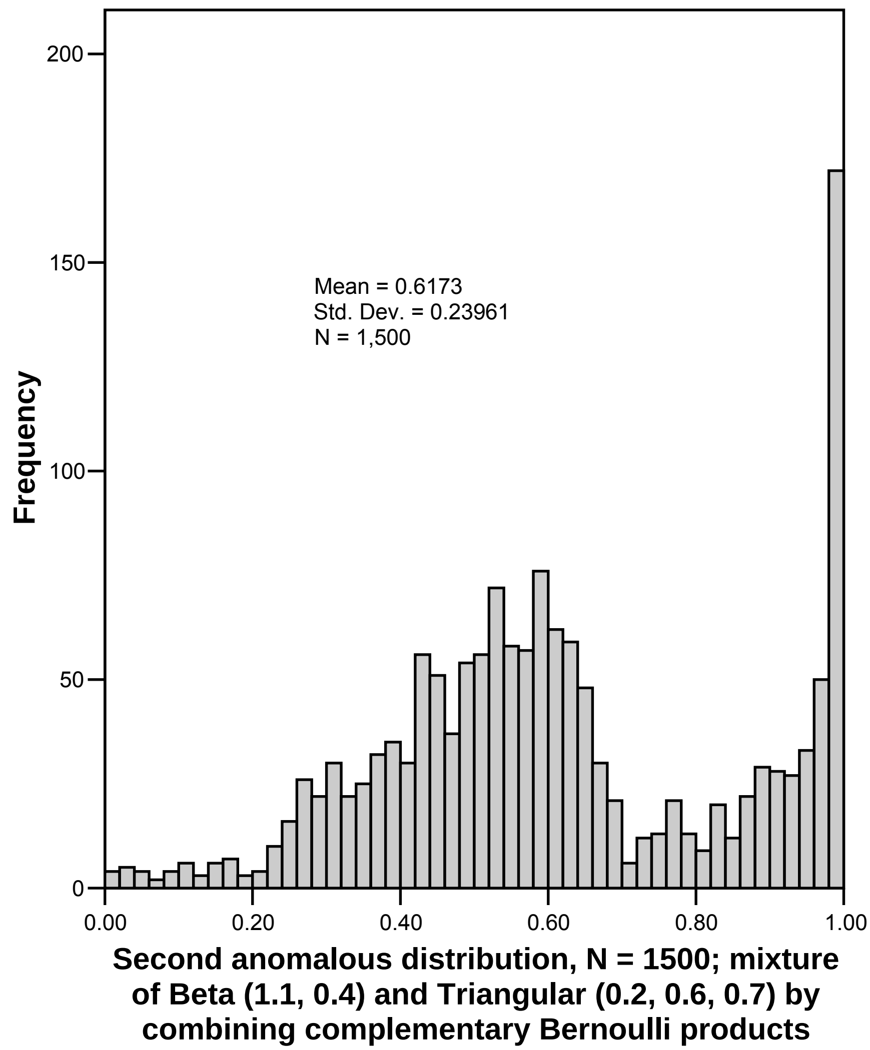

Figure 6. Histogram of alternative distribution no. 20 (a finite population of 1500 variates, as simulated in MINITAB; graph in SPSS.) The mixture was achieved by adding the product of a Bernoulli (p = 0.5) variate and a Beta (1.1, 0.4) variate to the product of the complement of the same Bernoulli variate and a Triangular (0.2, 0.6, 0.7) variate.

linked through the complementarity of b and (1 – b) and therefore dependent.

Then, $\mu_{X_1} = E[X_1] = \sum_{b=1,0} b \cdot f_b(b) \cdot \left(\int B \cdot f_B(B)\, dB \right) = \mu_b \cdot \mu_B = (0.5)(0.7\overline{3})$

$= 0.3\overline{6}$. Similarly, $\mu_{X_2} = E[X_2] = \sum_{b=1,0} (1-b) \cdot f_{1-b}(1-b) \cdot \left(\int T \cdot f_T(T)\, dT \right)$

$= \mu_{1-b} \cdot \mu_T = (0.5)(0.5) = 0.25$.

The theoretical expectations above can be compared to values obtained through analysis in MINITAB of components of the mixture. In addition to the statistics given above for the 1500 Beta and Triangular variates, those for the 1500 Bernoulli variates and their complements, respectively b_i and $1 - b_i$, along with those for the product components for this mixture variate are given in Table IV below, as computed in MINITAB.

Table IV

Statistics for Simulated Component Variates

variate or statistic	symbol	mean	variance or covariance
Bernoulli	b	0.4853	0.2500
Bernoulli complement	1 – b	0.5147	0.2500
Beta (1.1, 0.4)	B	0.73897	0.07683
Triangular(0.2, 0.6, 0.7)	T	0.49955	0.01243
product of Bernoulli and Beta	$X_1 = bB$	0.3602	0.1740
product of complement and Triangular	$X_2 = (1 - b)T$	0.25709	0.6880
covariance of X_1 and X_2	$s_{X_1 X_2}$		- 0.0926736

From this 'empirical' data, the mean of the second anomalous distribution as simulated is obtained as $\bar{x} \;=\; \bar{x}_b \cdot \bar{x}_B \;+\; \bar{x}_{1-b} \cdot \bar{x}_T \;=\; (0.4853)(0.73897)$ $+\; (0.5147)(0.49955) \;=\; 0.3586 + 0.2571 \;=\; 0.6157$. From the Monte Carlo procedure in this investigation, the average of sample means for 100,000 replications of this alternative distribution for each of the four sample sizes, as seen in Table 4, was 0.617, consistent with both the theoretical treatment and the empirical component analysis.

The variance of the Bernoulli mixture alternative distribution can be found as follows.

$$VAR\left(bB \;+\; (1-b)T\right) \;=\; E\left[\left(\left(bB \;+\; (1-b)T\right) \;-\; E\left[bB \;+\; (1-b)T\right]\right)^2\right] =$$

$$\sum_{b=0,1}\left\{\iint\left[\left(bB + (1-b)T\right) \;-\; \left(\tfrac{1}{2}\mu_B + \tfrac{1}{2}\mu_T\right)\right]^2 f_B(B)\, f_T(T)\, dB\, dT\right\} f_{b,1-b}\left(b,(1-b)\right) \;=$$

$$\sum_{b=1,0}\left\{\iint\left[\left(bB+(1-b)T\right)^2 - 2\left(bB+(1-b)T\right)\left(\tfrac{1}{2}\mu_B + \tfrac{1}{2}\mu_T\right) + \left(\tfrac{1}{2}\mu_B + \tfrac{1}{2}\mu_T\right)^2\right] f_B(B)\, f_T(T)\, dB\, dT\right\}$$

$$\bullet f_{b,1-b}\left(b,(1-b)\right) \;=$$ [33]

$$\tfrac{1}{2}\left\{\iint\left[B^2 \;-\; \mu_B B \;-\; \mu_T B \;+\; T^2 \;-\; \mu_B T \;-\; \mu_T T \;+\; \tfrac{1}{2}\mu_B^2 \;+\; \tfrac{1}{2}\mu_T^2 \;+\; \mu_B\mu_T\right] f_B(B)\, f_T(T)\, dB\, dT\right\} \;=$$

$$\tfrac{1}{2}\left\{\iint\left[\left(B^2 - 2\mu_B B + \mu_B^2\right) + \mu_B B - \tfrac{1}{2}\mu_B^2 + \left(T^2 - 2\mu_T T + \mu_T^2\right) + \mu_T T - \tfrac{1}{2}\mu_T^2 - \mu_T B - \mu_B T + \mu_B\mu_T\right]\right.$$

$$\left.\bullet f_B(B)\, f_T(T)\, dB\, dT\right\} \quad = \quad \tfrac{1}{2}\left\{\int\left[\int\left(B - \mu_B\right)^2 f_B(B)\, dB\right] f_T(T)\, dT \;+\right.$$

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[33] Where $E[b] \;=\; E[1-b] \;=\; \tfrac{1}{2}$.

$$\int\left[\int\left(T-\mu_T\right)^2 f_T\left(T\right)dT\right]f_B\left(B\right)dB \;+\; \mu_B\int\left[\int B\, f_B\left(B\right)dB\right]f_T\left(T\right)dT \;+$$

$$\mu_T\int\left[\int T\, f_T\left(T\right)dT\right]f_B\left(B\right)dB \;-\; \mu_T\int\left[\int B\, f_B\left(B\right)dB\right]f_T\left(T\right)dT \;-$$

$$\mu_B\int\left[\int T\, f_T\left(T\right)dT\right]f_B\left(B\right)dB \;+\; \left(-\tfrac{1}{2}\mu_B^2 - \tfrac{1}{2}\mu_T^2 + \mu_B\mu_T\right)\cdot\int\left[\int f_B\left(B\right)dB\right]f_T\left(T\right)dT\Big\}$$

$$= \tfrac{1}{2}\left\{\sigma_B^2 + \sigma_T^2 + \mu_B^2 + \mu_T^2 - \mu_T\mu_B - \mu_B\mu_T - \tfrac{1}{2}\mu_B^2 - \tfrac{1}{2}\mu_T^2 + \mu_B\mu_T\right\}$$

$$= \tfrac{1}{2}\sigma_B^2 + \tfrac{1}{2}\sigma_T^2 + \tfrac{1}{4}\mu_B^2 + \tfrac{1}{4}\mu_T^2 - \tfrac{1}{2}\mu_B\mu_T = \tfrac{1}{2}\sigma_B^2 + \tfrac{1}{2}\sigma_T^2 + \tfrac{1}{4}\left(\mu_B - \mu_T\right)^2.$$

Using the theoretical moments for the respective component distributions, the formula for the variance is evaluated as $VAR\left[b\,B + \left(1-b\right)T\right] = \tfrac{1}{2}\left(0.078\overline{2}\right) + \tfrac{1}{2}\left(0.009\overline{4}\right) + \tfrac{1}{4}\left(0.7\overline{3} - 0.5\right)^2 = 0.057\overline{4}$. Using the statistics for the component variates, the 'empirical' variance is evaluated as $s_X^2 = \overline{x}_b \cdot s_B^2 + \overline{x}_{1-b}\cdot s_T^2 + \overline{x}_b \cdot \overline{x}_{1-b} \bullet \left(\overline{x}_B - \overline{x}_T\right)^2 = \left(0.4853\right)\left(0.07683\right) + \left(0.5147\right)\left(0.01243\right) + \left(0.4853\right)\left(0.5147\right)\bullet\left(0.73897 - 0.49955\right)^2 = 0.05800.$ Alternatively, the empirical values can be analyzed, taking the variance of $X = X_1 + X_2$ in terms of the variances of the two product components, $X_1 = b\cdot B$ and $X_2 = \left(1-b\right)\cdot T$. As computed in MINITAB from the products of the respective 1500 variates, where the product components are subject to the interdependency of the Bernoulli variate and its complement, $s_X^2 = s_{X_1}^2 + s_{X_2}^2 + 2\cdot s_{X_1X_2} = 0.1740 + 0.06880 + 2\left(-0.0926736\right) = 0.05745$. Lastly, in the Monte Carlo procedure, the average of sample variances for 100,000 replications of this alternative distribution for each of the four sample sizes, as seen in Table 4, was 0.057, consistent with both the theoretical treatment and the empirical component analysis.

CHAPTER 4

RESULTS

Presentation of Monte Carlo Output

Sequence of Monte Carlo Programs and Output

The evidence from this study is exhibited by tabulated values generated during the execution of the researcher's FORTRAN programs embodying the conceptual framework and methodology for the simulations encompassed by the study. In order to proceed to presentation and discussion of that evidence, it will serve to introduce the sequence of the evidentiary Tables 1 – 5 as the output of Monte Carlo programs (which will be included as appendices) in Table V, below.

Table V

Summary of Tabular Output of the Simulation Study

Output Table	Information displayed in Table (for 100,000 replications in each case):	Outputting program and sequence in execution
Table 1	Critical values of MADMASR for n between 3 and 100	mcexptst01_outproc (2nd)
Table 2	Statistics of theta-star and MADMASR for n between 3 and 100	mcexpontest01 (1st)
Table 3	Type I error rates for MADMASR, testing four selected exponential distributions, at n = 4, 10, 20, and 50	mcexp_verify (3rd)
Table 4	Rejection rates (power) for MADMASR, for 20 alternative distributions, at n = 4, 10, 20, and 50	mcexp_rejectrates (4th)
Table 5	Power for 5 alternative tests, for 20 alternative distributions, at n = 4, 10, 20, and 50.	mcexptst_pwr (5th)

The Tabular Output

The researcher's FORTRAN programs were executed in succession several times during and subsequent to development of the Monte Carlo procedure, with varying but consistent results. The number of replications was set at 100,000, given that contemporary computing power for a desktop PC rendered that number feasible, with execution times for the most time-consuming, individual program not exceeding ten minutes. While independently run executions of the programs may not strongly misrepresent the results, the programs are linked, because some internal or intermediary outputs, not exhibited here, are used as inputs for subsequent programmed tasks.

In particular, the first of the FORTRAN programs, mcexpontest01, creates, in addition to Table 2, 42 files, each with 100,000 replicates of the test statistic, MADMASR, one for each sample size ranging between n = 3 and n = 100. These files are regenerated with each execution of mcexpontest01 and are stored on the researcher's computer hard drive. The second of the programs, mcexptst01_outproc, read and processed the 42 files one at a time, producing the critical values for MADMASR displayed in Table 1. At the same time, this program stored critical values for samples sized n = 4, 10, 20 and 50 in a file, saved on the hard drive, to be accessed by the subsequent FORTRAN programs which found Type I error rates and power (rejection rates for alternative distributions), mcexp_verify and mcexp_rejectrates, respectively.

The fifth of the programs simulates five competing goodness-of-fit tests, computing power for the four selected sample sizes and 20 alternative distributions. (Alternative number 14, simulated as Chi-Square with two degrees of freedom, is treated as an alternative distribution, for reasons of context treated above, although it is equivalent to an exponential distribution function.) All five of the foregoing FORTRAN programs utilize function or subroutine subprograms included in the researcher's utility subprogram module, utils4tstexpdist and in the RANGEN module (Blair, 1987).

The first table to be displayed, Table 1, shows critical values of the researcher's test statistic at four upper percentiles. This table is directly followed by the remaining four. Table 2 gives mean, standard deviation, skewness and kurtosis for the test statistic and for the exponential parameter estimate (theta-star), at each sample size. Table 3 shows Type I error rates upon testing samples simulated from exponential distributions with four distinct values of the parameter, for the four selected sample sizes. Tables 1 – 3 are each displayed on a single page.

Table 4 shows rejection rates for the four selected sample sizes for each of the 20 alternative distributions. Four alpha values are used here, 0.10, .0.05, 0.025, and 0.01. In addition means and standard deviations are given for the parameter estimate and for the generated samples, for each sample size and alternative distribution. The latter statistics helped verify that the procedures used to simulate samples from the various alternative distributions produced characteristics consistent with theory for those distributions. Table 4 was frequently the occasion for repeating an execution due to residual underflow manifesting itself in the researcher's numerical routines. The run represented here proceeded throughout the programming sequence without underflow.

Table 5 presents critical values for the Lilliefors-Kolmogorov-Smirnov, Cramer-von Mises, Shapiro-Wilk, Gini, and Kullback-Liebler statistics for the four selected sample sizes, at three upper percentiles, alpha = 0.1, 0.05, and 0.01. The critical values for the researcher's MADMASR statistic are included for comparison. After the critical values, Table 5 continues to show rejection rates in respect to those critical values, or power, for each selected sample size, for each alternative distribution. The rejection rates for the MADMASR statistic, here to three decimal places, are given for comparison to the five other tests. Table 5 is the longest table and is produced by the researcher's longest-running program in this study, mcexptst_pwr.

Tables 1 – 5 were produced in complete detail by force of the researcher's FORTRAN programming but have been reformatted to enhance their appearance and readability.

Table 1

Critical values of the MADMASR statistic (100,000 replications)

	upper percentiles			
sample size	*.900*	*.950*	*.975*	*.990*
3	1.6999	1.7168	1.7246	1.7290
4	1.5129	1.6446	1.7408	1.8342
5	2.0031	2.0795	2.1406	2.1978
6	2.1020	2.2151	2.2804	2.3389
7	2.3137	2.4026	2.4668	2.5305
8	2.4200	2.5067	2.5744	2.6424
9	2.5752	2.6649	2.7363	2.8085
10	2.7029	2.7873	2.8560	2.9284
11	2.8508	2.9383	3.0042	3.0752
12	2.9703	3.0549	3.1190	3.1859
13	3.0991	3.1890	3.2558	3.3242
14	3.2114	3.2995	3.3667	3.4331
15	3.3320	3.4223	3.4864	3.5518
16	3.4367	3.5304	3.5960	3.6610
17	3.5444	3.6431	3.7131	3.7793
18	3.6488	3.7492	3.8166	3.8823
19	3.7512	3.8562	3.9299	4.0006
20	3.8537	3.9575	4.0273	4.0978
21	3.9471	4.0586	4.1338	4.2049
22	4.0377	4.1546	4.2287	4.3024
23	4.1380	4.2511	4.3295	4.4008
24	4.2280	4.3476	4.4227	4.4947
25	4.3135	4.4366	4.5171	4.5877
26	4.3985	4.5243	4.6048	4.6768
27	4.4822	4.6081	4.6917	4.7654
28	4.5679	4.6974	4.7780	4.8543
29	4.6503	4.7817	4.8665	4.9458
30	4.7269	4.8633	4.9494	5.0352
35	5.1053	5.2540	5.3484	5.4359
40	5.4563	5.6172	5.7168	5.8074
45	5.7776	5.9490	6.0596	6.1563
50	6.0733	6.2599	6.3780	6.4831
55	6.3597	6.5525	6.6792	6.7847
60	6.6357	6.8441	6.9733	7.0903
65	6.8907	7.1098	7.2508	7.3719
70	7.1412	7.3735	7.5155	7.6438
75	7.3816	7.6202	7.7708	7.9045
80	7.6157	7.8558	8.0186	8.1596
85	7.8520	8.0942	8.2570	8.4036
90	8.0591	8.3145	8.4794	8.6309
95	8.2737	8.5310	8.7046	8.8662
100	8.4821	8.7404	8.9180	9.0864

Table 2

Statistics by sample size for 100,000 replications of theta-star and MADMASR

sample size	parameter estimate				MADMASR statistic			
	mean	std. dev.	skew-ness	kurtosis	mean	std. dev.	skew-ness	kurtosis
3	1.057	0.636	1.310	2.824	1.450	0.213	-0.489	-0.990
4	1.034	0.540	1.103	1.870	1.177	0.218	1.104	0.987
5	1.024	0.482	1.025	1.627	1.667	0.251	0.066	-0.787
6	1.015	0.438	0.944	1.450	1.755	0.270	0.037	-0.690
7	1.012	0.404	0.884	1.327	1.892	0.314	-0.006	-0.784
8	1.008	0.377	0.841	1.166	2.001	0.325	-0.148	-0.703
9	1.007	0.357	0.782	0.981	2.136	0.348	-0.206	-0.715
10	1.005	0.339	0.760	0.963	2.240	0.370	-0.277	-0.708
11	1.002	0.321	0.714	0.876	2.350	0.401	-0.310	-0.707
12	1.001	0.309	0.673	0.702	2.443	0.424	-0.346	-0.703
13	1.002	0.299	0.689	0.796	2.541	0.449	-0.360	-0.687
14	1.001	0.285	0.647	0.817	2.626	0.470	-0.376	-0.668
15	1.000	0.277	0.632	0.619	2.716	0.493	-0.381	-0.665
16	1.001	0.268	0.602	0.608	2.797	0.511	-0.374	-0.665
17	1.000	0.262	0.608	0.631	2.876	0.531	-0.353	-0.680
18	0.999	0.253	0.586	0.631	2.951	0.548	-0.334	-0.685
19	0.999	0.248	0.585	0.583	3.026	0.568	-0.317	-0.689
20	0.999	0.241	0.542	0.466	3.102	0.583	-0.297	-0.707
21	0.998	0.235	0.535	0.515	3.167	0.600	-0.273	-0.711
22	1.000	0.230	0.527	0.468	3.237	0.613	-0.255	-0.709
23	0.999	0.226	0.519	0.456	3.307	0.630	-0.236	-0.717
24	0.998	0.220	0.507	0.455	3.375	0.646	-0.225	-0.723
25	1.000	0.216	0.498	0.438	3.437	0.659	-0.212	-0.705
26	0.999	0.212	0.493	0.412	3.500	0.672	-0.201	-0.704
27	1.000	0.209	0.469	0.353	3.565	0.684	-0.199	-0.688
28	0.998	0.204	0.500	0.472	3.627	0.699	-0.185	-0.704
29	0.999	0.201	0.457	0.300	3.688	0.712	-0.178	-0.700
30	0.998	0.198	0.461	0.387	3.749	0.728	-0.193	-0.676
35	0.998	0.184	0.450	0.380	4.036	0.788	-0.170	-0.675
40	0.998	0.171	0.407	0.291	4.308	0.847	-0.180	-0.678
45	0.998	0.162	0.387	0.279	4.557	0.904	-0.200	-0.671
50	0.999	0.155	0.390	0.284	4.793	0.956	-0.219	-0.660
55	0.998	0.147	0.357	0.215	5.016	1.006	-0.224	-0.693
60	0.998	0.141	0.332	0.196	5.231	1.061	-0.234	-0.704
65	0.999	0.136	0.335	0.246	5.435	1.108	-0.245	-0.710
70	0.998	0.130	0.316	0.180	5.643	1.150	-0.257	-0.723
75	0.999	0.126	0.316	0.225	5.828	1.199	-0.264	-0.728
80	0.998	0.123	0.282	0.137	6.005	1.246	-0.257	-0.764
85	0.999	0.119	0.296	0.131	6.191	1.291	-0.274	-0.750
90	0.998	0.115	0.282	0.150	6.352	1.328	-0.263	-0.769
95	0.999	0.112	0.254	0.131	6.516	1.365	-0.258	-0.775
100	0.998	0.110	0.266	0.125	6.667	1.411	-0.253	-0.802

Table 3

Type I error rates for 100,000 replications of the MADMASR goodness-of-fit statistic with simulated random sampling from exponential distributions with four selected parameters, for four selected sample sizes each, at four selected right-tailed critical values of the statistic. The mean and standard deviation of the estimated exponential parameter for all replications are also shown for each case.

		alpha level (right tail)				theta-star estimate	
pop. theta	sample size	.10	.05	.025	.01	mean	std. dev.
0.1							
	4	0.1005	0.0504	0.0258	0.0101	0.103	0.054
	10	0.0990	0.0497	0.0249	0.0097	0.101	0.034
	20	0.0991	0.0497	0.0254	0.0101	0.100	0.024
	50	0.1028	0.0525	0.0255	0.0099	0.100	0.015
0.5							
	4	0.0993	0.0504	0.0261	0.0098	0.516	0.272
	10	0.0988	0.0501	0.0251	0.0095	0.502	0.169
	20	0.0997	0.0509	0.0259	0.0101	0.499	0.120
	50	0.1020	0.0518	0.0257	0.0100	0.499	0.077
2.0							
	4	0.1009	0.0504	0.0257	0.0101	2.059	1.077
	10	0.0974	0.0494	0.0244	0.0099	2.009	0.679
	20	0.1001	0.0503	0.0252	0.0102	1.998	0.482
	50	0.1002	0.0494	0.0237	0.0090	1.996	0.307
10.0							
	4	0.0992	0.0507	0.0261	0.0103	10.304	5.379
	10	0.0994	0.0510	0.0246	0.0096	10.048	3.380
	20	0.0987	0.0504	0.0263	0.0105	9.989	2.407
	50	0.1010	0.0507	0.0243	0.0097	9.989	1.540

Table 4

Rejection rates for 100,000 replications of the MADMASR goodness-of-fit statistic with simulated random sampling from 20 alternative distributions, for four selected sample sizes each, at four selected right-tailed critical values of the statistic. The mean and standard deviation of the estimated exponential parameter and the average sample mean and sample variance for all replications are also shown for each case.

sample	alpha level (right tail)				theta-star (est'd exp. parameter)		all samples	
size	*.10*	*.05*	*.025*	*.01*	mean	std. dev.	avg. mean	avg. var.
alt. dist. no. 1: mixed exponential (θ = 0.5) and exponential (θ = 4.0)								
4	0.0759	0.0362	0.0179	0.0068	2.619	2.014	2.256	11.320
10	0.0390	0.0181	0.0087	0.0033	2.649	1.284	2.251	11.189
20	0.0141	0.0050	0.0020	0.0006	2.684	0.919	2.249	11.163
50	0.0058	0.0016	0.0004	0.0001	2.724	0.591	2.251	11.192
alt. dist. no. 2: mixed exponential (θ = 1.0) and normal (μ = 10, σ = 1)								
4	0.0230	0.0100	0.0046	0.0016	5.374	1.546	5.492	21.203
10	0.1607	0.0316	0.0053	0.0008	4.996	0.724	5.505	21.262
20	0.0751	0.0026	0.0002	0.0000	4.823	0.461	5.497	21.236
50	0.0000	0.0000	0.0000	0.0000	4.701	0.270	5.502	21.248
alt. dist. no. 3: gamma (σ = 0.625, λ = 1.6)								
4	0.1046	0.0538	0.0282	0.0109	0.962	0.403	0.999	0.622
10	0.1079	0.0542	0.0265	0.0097	0.919	0.249	0.999	0.624
20	0.1295	0.0659	0.0336	0.0128	0.907	0.175	1.000	0.625
50	0.1360	0.0669	0.0314	0.0110	0.899	0.111	0.999	0.624
alt. dist. no. 4: gamma (σ = 0.25, λ = 4.0)								
4	0.0998	0.0531	0.0272	0.0108	0.860	0.232	1.001	0.251
10	0.1081	0.0516	0.0243	0.0084	0.793	0.139	0.999	0.249
20	0.1426	0.0720	0.0351	0.0135	0.772	0.096	1.000	0.250
50	0.0718	0.0218	0.0071	0.0018	0.757	0.061	1.000	0.250

Table 4 – continued

sample	alpha level (right tail)				theta-star (est'd exp. parameter)		all samples	
size	*.10*	*.05*	*.025*	*.01*	mean	std. dev.	avg. mean	avg. var.
alt. dist. no. 5: Weibull ($\sigma = 0.5$, $\lambda = 0.5$)								
4	0.0606	0.0287	0.0138	0.0052	1.243	1.479	0.996	4.878
10	0.0245	0.0111	0.0052	0.0020	1.315	1.043	1.002	5.068
20	0.0053	0.0020	0.0007	0.0002	1.361	0.778	1.000	4.985
50	0.0007	0.0002	0.0001	0.0000	1.416	0.529	1.001	4.998
alt. dist. no. 6: Weibull ($\sigma = 1.10773$, $\lambda = 1.5$)								
4	0.1015	0.0532	0.0279	0.0108	0.928	0.318	1.000	0.459
10	0.1108	0.0537	0.0255	0.0090	0.874	0.191	1.000	0.461
20	0.1654	0.0866	0.0452	0.0179	0.856	0.133	0.999	0.461
50	0.1463	0.0607	0.0248	0.0080	0.844	0.083	1.000	0.460
alt. dist. no. 7: Weibull ($\sigma = 1.128387$, $\lambda = 2.0$)								
4	0.0936	0.0499	0.0264	0.0106	0.870	0.224	1.000	0.272
10	0.1131	0.0533	0.0242	0.0086	0.803	0.131	1.000	0.273
20	0.1735	0.0897	0.0450	0.0177	0.781	0.089	1.000	0.273
50	0.0641	0.0161	0.0041	0.0009	0.765	0.055	1.000	0.273
alt. dist. no. 8: Gumbel ($\mu = 0.42278$, $\sigma = 1.0$)								
4	0.1022	0.0535	0.0284	0.0111	1.159	0.608	1.000	1.650
10	0.1058	0.0515	0.0244	0.0091	1.136	0.369	1.000	1.640
20	0.1187	0.0575	0.0278	0.0107	1.132	0.261	1.000	1.648
50	0.0450	0.0126	0.0039	0.0010	1.129	0.165	1.000	1.644

Table 4 – continued

sample	alpha level (right tail)				theta-star (est'd exp. parameter)		all samples	
size	*.10*	*.05*	*.025*	*.01*	mean	std. dev.	avg. mean	avg. var.
alt. dist. no. 9: lognormal (x; μ = 0.0, σ = 0.2) where μ = mean(ln(x)) and σ = sd(ln(x))								
4	0.0924	0.0492	0.0255	0.0103	0.758	0.089	1.020	0.043
10	0.1078	0.0517	0.0235	0.0081	0.674	0.052	1.020	0.042
20	0.1306	0.0617	0.0303	0.0118	0.645	0.036	1.020	0.043
50	0.0219	0.0047	0.0012	0.0003	0.625	0.023	1.020	0.042
alt. dist. no. 10: lognormal (μ = 0.0, σ = 1.0)								
4	0.0977	0.0486	0.0250	0.0094	1.744	1.368	1.649	4.657
10	0.0850	0.0420	0.0207	0.0081	1.734	0.931	1.648	4.649
20	0.0452	0.0214	0.0107	0.0042	1.752	0.706	1.648	4.701
50	0.0217	0.0098	0.0044	0.0017	1.780	0.475	1.649	4.654
alt. dist. no. 11: lognormal (μ = 0.0 , σ = 2.0)								
4	0.0593	0.0280	0.0140	0.0055	9.837	38.261	7.322	2648.960
10	0.0203	0.0096	0.0047	0.0015	10.852	31.230	7.345	3216.050
20	0.0026	0.0009	0.0003	0.0001	11.776	22.495	7.404	2629.720
50	0.0000	0.0000	0.0000	0.0000	12.824	16.311	7.412	2560.620
alt. dist. no. 12: Makeham (ξ = 1.0, λ = 1.0 , θ = 1.0) with DLMF parameterization								
4	0.0985	0.0512	0.0265	0.0104	0.399	0.161	0.404	0.112
10	0.1073	0.0522	0.0249	0.0094	0.381	0.096	0.404	0.113
20	0.1684	0.0919	0.0496	0.0203	0.375	0.066	0.404	0.113
50	0.2271	0.1178	0.0579	0.0228	0.371	0.041	0.404	0.112

Table 4 – continued

sample	alpha level (right tail)				theta-star (est'd exp. parameter)		all samples	
size	*.10*	*.05*	*.025*	*.01*	mean	std. dev.	avg. mean	avg. var.
alt. dist. no. 13: chi-square, df = 1								
4	0.0804	0.0393	0.0199	0.0074	1.146	0.833	1.002	1.999
10	0.0569	0.0277	0.0129	0.0052	1.154	0.540	1.002	2.009
20	0.0316	0.0131	0.0058	0.0019	1.165	0.387	1.000	1.997
50	0.0184	0.0066	0.0023	0.0006	1.180	0.252	1.000	2.002
alt. dist. no. 14: chi-square, df = 2 [exp(θ = 2.0)]								
4	0.1002	0.0509	0.0259	0.0104	2.065	1.081	1.999	4.011
10	0.0995	0.0503	0.0245	0.0095	2.011	0.675	2.001	4.008
20	0.0975	0.0493	0.0254	0.0100	2.001	0.481	2.001	4.006
50	0.0993	0.0500	0.0242	0.0096	1.997	0.306	2.000	3.998
alt. dist. no. 15: chi-square, df = 3								
4	0.1032	0.0525	0.0270	0.0100	2.914	1.256	2.997	5.979
10	0.1085	0.0549	0.0269	0.0103	2.794	0.775	3.001	6.012
20	0.1269	0.0660	0.0338	0.0134	2.754	0.549	2.998	5.997
50	0.1363	0.0673	0.0322	0.0123	2.735	0.349	3.000	5.999
alt. dist. no. 16: chi-square, df = 4								
4	0.1049	0.0549	0.0288	0.0112	3.742	1.411	4.004	8.034
10	0.1097	0.0551	0.0264	0.0104	3.542	0.862	4.005	8.040
20	0.1397	0.0725	0.0363	0.0146	3.471	0.604	3.996	7.988
50	0.1300	0.0569	0.0248	0.0080	3.435	0.384	4.000	8.002

Table 4 – continued

sample size	alpha level (right tail) .10	.05	.025	.01	theta-star (est'd exp. parameter) mean	std. dev.	all samples avg. mean	avg. var.
alt. dist. no. 17: uniform on (0.0, 1.0)								
4	0.0681	0.0352	0.0182	0.0070	0.447	0.100	0.500	0.083
10	0.2524	0.1242	0.0558	0.0179	0.410	0.051	0.500	0.083
20	0.5412	0.3151	0.1734	0.0735	0.396	0.033	0.500	0.083
50	0.0684	0.0077	0.0012	0.0001	0.385	0.019	0.500	0.083
alt. dist. no. 18: half-normal (μ = 0.0, σ = 1.0 for the assoc. normal dist.)								
4	0.0965	0.0501	0.0264	0.0103	0.766	0.280	0.798	0.364
10	0.1123	0.0527	0.0242	0.0085	0.727	0.165	0.798	0.364
20	0.1924	0.1064	0.0571	0.0239	0.713	0.114	0.798	0.364
50	0.2099	0.0989	0.0458	0.0165	0.703	0.071	0.798	0.363
alt. dist. no. 19: $X = 0.5 \cdot Beta$ (λ_1 = 1.1, λ_2 = 0.4) + 0.5·Triangular (a = 0.2, M =0.6, b = 0.7)								
4	0.0449	0.0237	0.0126	0.0052	0.466	0.040	0.619	0.022
10	0.3016	0.1588	0.0757	0.0265	0.412	0.020	0.619	0.022
20	0.2418	0.0921	0.0370	0.0114	0.393	0.012	0.619	0.022
50	0.0000	0.0000	0.0000	0.0000	0.379	0.007	0.619	0.022
alt. dist. no. 20: X = Bernoulli (p = .5)·Beta (λ_1 = 1.1, λ_2 = 0.4) + (1 – Bernoulli)·Triangular (a = 0.2, M =0.6, b = 0.7)								
4	0.0935	0.0508	0.0271	0.0110	0.504	0.089	0.618	0.057
10	0.2501	0.1149	0.0477	0.0147	0.456	0.046	0.617	0.057
20	0.5370	0.2932	0.1384	0.0463	0.438	0.029	0.617	0.057
50	0.0250	0.0015	0.0002	0.0000	0.425	0.017	0.617	0.057

Table 5

Comparison of Various Goodness-of-fit Statistics, testing 20 Alternative Distributions for Exponentiality

ks	Lilliefors-Kolmogorov-Smirnov statistic (Lilliefors, 1969)
cvm	Cramer-von Mises statistic with K transformation (Seshadri, Csorgo, & Stephens, 1969)
sw	Shapiro-WIlk statistic (Metz, Haccou, & Meelis, 1994)
gini	Gini statistic (Gail & Gastwirth, 1978)
klc	Kullback-Liebler information statistic (except for Gumbel alternative; Choi, Kim, & Song, 2004)
MADMASR	Maximum Absolute Deviation from the Median Absolute Standardized Residual

		Critical Values for Each Test Under Null Hypothesis of Exponentiality					
sig. level (right tail)	*sample size*	*ks*	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alpha = .10	4	0.444	0.604	0.751	0.749	0.998	1.513
	10	0.297	0.520	0.220	0.624	0.992	2.703
	20	0.213	0.442	0.089	0.589	1.024	3.854
	50	0.137	0.388	0.029	0.553	1.028	6.073
alpha = .05	4	0.484	0.752	0.839	0.809	1.120	1.645
	10	0.324	0.666	0.258	0.658	1.032	2.787
	20	0.234	0.583	0.101	0.612	1.051	3.958
	50	0.151	0.513	0.032	0.567	1.043	6.260
alpha = .01	4	0.557	0.984	0.969	0.912	1.344	1.834
	10	0.381	0.989	0.347	0.720	1.107	2.958
	20	0.277	0.907	0.128	0.657	1.100	4.098
	50	0.179	0.829	0.037	0.596	1.069	6.483

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 1:		*mixed exponential (θ = 0.5) and exponential (θ = 4.0)*					
alpha = .10	4	0.163	0.052	0.052	0.301	0.307	0.076
	10	0.384	0.051	0.009	0.594	0.156	0.039
	20	0.663	0.367	0.001	0.830	0.059	0.014
	50	0.960	0.933	0.000	0.991	0.004	0.006
alpha = .05	4	0.096	0.025	0.026	0.196	0.211	0.036
	10	0.288	0.020	0.004	0.464	0.080	0.018
	20	0.557	0.241	0.000	0.732	0.028	0.005
	50	0.929	0.881	0.000	0.977	0.002	0.002
alpha = .01	4	0.043	0.005	0.005	0.057	0.084	0.007
	10	0.140	0.002	0.000	0.232	0.016	0.003
	20	0.347	0.081	0.000	0.491	0.005	0.001
	50	0.817	0.700	0.000	0.908	0.000	0.000

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 2:		*mixed exponential (θ = 1.0) and normal (μ = 10, σ = 1)*					
alpha = .10	4	0.308	0.275	0.275	0.201	0.203	0.023
	10	0.721	0.580	0.233	0.147	0.010	0.161
	20	0.982	0.963	0.305	0.095	0.000	0.075
	50	1.000	1.000	0.483	0.048	0.000	0.000
alpha = .05	4	0.123	0.258	0.258	0.140	0.150	0.010
	10	0.528	0.396	0.183	0.089	0.004	0.032
	20	0.939	0.871	0.204	0.055	0.000	0.003
	50	1.000	1.000	0.349	0.025	0.000	0.000
alpha = .01	4	0.081	0.155	0.156	0.048	0.055	0.002
	10	0.211	0.191	0.072	0.023	0.001	0.001
	20	0.693	0.490	0.083	0.017	0.000	0.000
	50	1.000	0.999	0.156	0.006	0.000	0.000

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 3:		*gamma (σ = 0.625, λ = 1.6)*					
alpha = .10	4	0.152	0.128	0.127	0.027	0.030	0.105
	10	0.214	0.203	0.206	0.007	0.019	0.108
	20	0.331	0.323	0.328	0.002	0.012	0.130
	50	0.627	0.607	0.587	0.000	0.005	0.136
alpha = .05	4	0.082	0.063	0.064	0.010	0.010	0.054
	10	0.124	0.114	0.117	0.002	0.008	0.054
	20	0.214	0.203	0.205	0.001	0.005	0.066
	50	0.483	0.472	0.445	0.000	0.002	0.067
alpha = .01	4	0.014	0.013	0.013	0.001	0.001	0.011
	10	0.030	0.028	0.029	0.000	0.001	0.010
	20	0.065	0.063	0.063	0.000	0.001	0.013
	50	0.224	0.220	0.208	0.000	0.000	0.011

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 4:		*gamma (σ = 0.25, λ = 4.0)*					
alpha = .10	4	0.422	0.173	0.173	0.000	0.000	0.100
	10	0.813	0.424	0.431	0.000	0.000	0.108
	20	0.987	0.751	0.752	0.000	0.000	0.143
	50	1.000	0.989	0.987	0.000	0.000	0.072
alpha = .05	4	0.271	0.089	0.089	0.000	0.000	0.053
	10	0.669	0.284	0.292	0.000	0.000	0.052
	20	0.961	0.624	0.621	0.000	0.000	0.072
	50	1.000	0.977	0.970	0.000	0.000	0.022
alpha = .01	4	0.057	0.018	0.018	0.000	0.000	0.011
	10	0.330	0.097	0.103	0.000	0.000	0.008
	20	0.801	0.354	0.350	0.000	0.000	0.014
	50	1.000	0.915	0.898	0.000	0.000	0.002

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 5:		*Weibull ($\sigma = 0.5$, $\lambda = 0.5$)*					
alpha = .10	4	0.314	0.038	0.038	0.476	0.451	0.061
	10	0.656	0.185	0.004	0.825	0.194	0.025
	20	0.913	0.736	0.000	0.971	0.041	0.005
	50	0.999	0.998	0.000	1.000	0.000	0.001
alpha = .05	4	0.203	0.018	0.019	0.361	0.348	0.029
	10	0.570	0.107	0.001	0.743	0.115	0.011
	20	0.867	0.630	0.000	0.945	0.023	0.002
	50	0.998	0.995	0.000	1.000	0.000	0.000
alpha = .01	4	0.108	0.004	0.004	0.168	0.194	0.005
	10	0.384	0.027	0.000	0.547	0.032	0.002
	20	0.732	0.417	0.000	0.856	0.006	0.000
	50	0.992	0.979	0.000	0.998	0.000	0.000

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 6:		*Weibull (σ = 1.10773, λ = 1.5)*					
alpha = .10	4	0.194	0.154	0.154	0.015	0.013	0.102
	10	0.334	0.316	0.324	0.001	0.007	0.111
	20	0.546	0.563	0.580	0.000	0.002	0.165
	50	0.898	0.920	0.930	0.000	0.000	0.146
alpha = .05	4	0.110	0.078	0.078	0.005	0.004	0.053
	10	0.211	0.194	0.200	0.000	0.002	0.054
	20	0.395	0.411	0.421	0.000	0.001	0.087
	50	0.811	0.852	0.858	0.000	0.000	0.061
alpha = .01	4	0.020	0.016	0.016	0.001	0.000	0.011
	10	0.061	0.055	0.058	0.000	0.000	0.009
	20	0.159	0.166	0.169	0.000	0.000	0.018
	50	0.549	0.621	0.630	0.000	0.000	0.008

Table 5 – continued

sig. level (right tail)	sample size	Rejection Rates for 20 Alternative Distributions					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 7:		*Weibull (σ = 1.128387, λ = 2.0)*					
alpha = .10	4	0.346	0.192	0.193	0.002	0.001	0.094
	10	0.669	0.509	0.522	0.000	0.000	0.113
	20	0.928	0.845	0.863	0.000	0.000	0.174
	50	1.000	0.998	0.999	0.000	0.000	0.064
alpha = .05	4	0.219	0.100	0.100	0.000	0.000	0.050
	10	0.510	0.356	0.369	0.000	0.000	0.053
	20	0.850	0.735	0.755	0.000	0.000	0.090
	50	0.999	0.995	0.997	0.000	0.000	0.016
alpha = .01	4	0.045	0.021	0.021	0.000	0.000	0.011
	10	0.215	0.131	0.139	0.000	0.000	0.009
	20	0.581	0.461	0.476	0.000	0.000	0.018
	50	0.988	0.968	0.976	0.000	0.000	0.001

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 8:		*Gumbel (μ = 0.42278, σ = 1.0)*					
alpha = .10	4	0.578	0.175	0.175	0.525	0.000	0.102
	10	0.844	0.448	0.451	0.637	0.000	0.106
	20	0.976	0.774	0.759	0.779	0.000	0.119
	50	1.000	0.992	0.983	0.947	0.000	0.045
alpha = .05	4	0.532	0.090	0.090	0.469	0.000	0.054
	10	0.815	0.312	0.316	0.576	0.000	0.052
	20	0.969	0.663	0.643	0.724	0.000	0.058
	50	1.000	0.983	0.967	0.924	0.000	0.013
alpha = .01	4	0.475	0.019	0.019	0.386	0.000	0.011
	10	0.770	0.117	0.122	0.470	0.000	0.009
	20	0.956	0.418	0.399	0.612	0.000	0.011
	50	1.000	0.938	0.905	0.860	0.000	0.001

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 9:		*lognormal (x; μ = 0.0, σ = 0.2)*		*where μ = mean(ln(x)) and σ = sd(ln(x))*			
alpha = .10	4	0.990	0.208	0.208	0.000	0.000	0.092
	10	1.000	0.582	0.591	0.000	0.000	0.108
	20	1.000	0.900	0.904	0.000	0.000	0.131
	50	1.000	0.999	0.999	0.000	0.000	0.022
alpha = .05	4	0.942	0.109	0.110	0.000	0.000	0.049
	10	1.000	0.437	0.449	0.000	0.000	0.052
	20	1.000	0.827	0.831	0.000	0.000	0.062
	50	1.000	0.999	0.998	0.000	0.000	0.005
alpha = .01	4	0.495	0.023	0.023	0.000	0.000	0.010
	10	1.000	0.194	0.203	0.000	0.000	0.008
	20	1.000	0.619	0.620	0.000	0.000	0.012
	50	1.000	0.991	0.991	0.000	0.000	0.000

Table 5 – continued

		Rejection Rates for 20 Alternative Distributions					
sig. level (right tail)	*sample size*	*ks*	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 10:		*lognormal (μ = 0.0, σ = 1.0)*					
alpha = .10	4	0.127	0.085	0.085	0.082	0.110	0.098
	10	0.169	0.079	0.067	0.132	0.043	0.085
	20	0.224	0.120	0.047	0.169	0.027	0.045
	50	0.381	0.257	0.022	0.233	0.007	0.022
alpha = .05	4	0.069	0.042	0.042	0.038	0.058	0.049
	10	0.099	0.038	0.033	0.082	0.017	0.042
	20	0.138	0.068	0.024	0.112	0.010	0.021
	50	0.253	0.179	0.011	0.165	0.002	0.010
alpha = .01	4	0.014	0.008	0.008	0.004	0.013	0.009
	10	0.030	0.007	0.007	0.030	0.002	0.008
	20	0.048	0.021	0.005	0.047	0.001	0.004
	50	0.100	0.081	0.002	0.077	0.000	0.002

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 11: lognormal (μ = 0.0 , σ = 2.0)							
alpha = .10	4	0.311	0.038	0.038	0.473	0.465	0.059
	10	0.688	0.315	0.004	0.834	0.131	0.020
	20	0.928	0.833	0.000	0.975	0.022	0.003
	50	0.999	0.999	0.000	1.000	0.000	0.000
alpha = .05	4	0.215	0.018	0.018	0.364	0.367	0.028
	10	0.617	0.227	0.002	0.765	0.071	0.010
	20	0.895	0.767	0.000	0.958	0.011	0.001
	50	0.999	0.998	0.000	1.000	0.000	0.000
alpha = .01	4	0.136	0.004	0.004	0.176	0.217	0.006
	10	0.468	0.101	0.000	0.608	0.017	0.002
	20	0.805	0.621	0.000	0.898	0.002	0.000
	50	0.995	0.993	0.000	0.999	0.000	0.000

Table 5 – continued

sig. level (right tail)	sample size	Rejection Rates for 20 Alternative Distributions					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 12:		*Makeham (ξ = 1.0, λ = 1.0 , θ = 1.0) with DLMF parameterization*					
alpha = .10	4	0.115	0.124	0.124	0.067	0.058	0.099
	10	0.130	0.180	0.182	0.029	0.044	0.107
	20	0.157	0.258	0.280	0.014	0.031	0.168
	50	0.258	0.434	0.522	0.004	0.019	0.227
alpha = .05	4	0.059	0.062	0.062	0.033	0.026	0.051
	10	0.067	0.098	0.099	0.011	0.020	0.052
	20	0.084	0.148	0.159	0.005	0.013	0.092
	50	0.148	0.292	0.356	0.002	0.008	0.118
alpha = .01	4	0.011	0.012	0.012	0.007	0.004	0.010
	10	0.013	0.023	0.023	0.001	0.003	0.009
	20	0.017	0.036	0.037	0.000	0.002	0.020
	50	0.033	0.091	0.121	0.000	0.001	0.023

Table 5 – continued

sig. level (right tail)	*sample size*	*ks*	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
		Rejection Rates for 20 Alternative Distributions					
alternative distribution no. 13:	*chi-square, df = 1*						
alpha = .10	4	0.186	0.061	0.062	0.320	0.300	0.080
	10	0.344	0.047	0.018	0.557	0.249	0.057
	20	0.583	0.250	0.004	0.763	0.136	0.032
	50	0.912	0.820	0.000	0.968	0.014	0.018
alpha = .05	4	0.103	0.030	0.030	0.217	0.205	0.039
	10	0.257	0.019	0.007	0.431	0.147	0.028
	20	0.470	0.152	0.001	0.654	0.078	0.013
	50	0.856	0.728	0.000	0.937	0.007	0.007
alpha = .01	4	0.043	0.006	0.006	0.079	0.088	0.007
	10	0.119	0.002	0.001	0.219	0.042	0.005
	20	0.271	0.048	0.000	0.419	0.021	0.002
	50	0.691	0.492	0.000	0.820	0.002	0.001

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 14: chi-square, df = 2 [exp(θ = 2.0)]							
alpha = .10	4	0.102	0.101	0.101	0.103	0.101	0.100
	10	0.099	0.100	0.099	0.100	0.101	0.100
	20	0.100	0.101	0.101	0.100	0.100	0.098
	50	0.101	0.099	0.099	0.101	0.098	0.099
alpha = .05	4	0.050	0.050	0.050	0.053	0.052	0.051
	10	0.050	0.051	0.050	0.050	0.050	0.050
	20	0.050	0.050	0.050	0.050	0.050	0.049
	50	0.050	0.050	0.050	0.051	0.049	0.050
alpha = .01	4	0.010	0.010	0.010	0.010	0.010	0.010
	10	0.010	0.010	0.010	0.010	0.010	0.010
	20	0.009	0.010	0.010	0.010	0.010	0.010
	50	0.010	0.009	0.010	0.010	0.010	0.010

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 15: chi-square, df = 3							
alpha = .10	4	0.139	0.124	0.124	0.033	0.036	0.103
	10	0.189	0.187	0.191	0.011	0.026	0.109
	20	0.275	0.287	0.293	0.004	0.019	0.127
	50	0.517	0.522	0.511	0.000	0.010	0.136
alpha = .05	4	0.075	0.063	0.063	0.013	0.013	0.053
	10	0.107	0.104	0.107	0.004	0.011	0.055
	20	0.169	0.177	0.181	0.001	0.007	0.066
	50	0.374	0.388	0.370	0.000	0.004	0.067
alpha = .01	4	0.012	0.012	0.012	0.002	0.001	0.010
	10	0.025	0.025	0.025	0.000	0.001	0.010
	20	0.048	0.051	0.051	0.000	0.001	0.013
	50	0.152	0.165	0.157	0.000	0.000	0.012

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 16:	*chi-square, df = 4*						
alpha = .10	4	0.196	0.139	0.140	0.012	0.014	0.105
	10	0.340	0.260	0.266	0.001	0.006	0.110
	20	0.562	0.457	0.458	0.000	0.002	0.140
	50	0.912	0.828	0.800	0.000	0.000	0.130
alpha = .05	4	0.111	0.071	0.070	0.004	0.004	0.055
	10	0.214	0.155	0.159	0.000	0.002	0.055
	20	0.410	0.316	0.314	0.000	0.001	0.073
	50	0.831	0.728	0.686	0.000	0.000	0.057
alpha = .01	4	0.020	0.014	0.014	0.000	0.000	0.011
	10	0.063	0.042	0.044	0.000	0.000	0.010
	20	0.165	0.118	0.115	0.000	0.000	0.015
	50	0.578	0.466	0.427	0.000	0.000	0.008

Table 5 – continued

		Rejection Rates for 20 Alternative Distributions					
sig. level (right tail)	*sample size*	*ks*	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 17:		*uniform on (0.0, 1.0)*					
alpha = .10	4	0.227	0.253	0.253	0.023	0.012	0.068
	10	0.420	0.622	0.627	0.001	0.001	0.252
	20	0.682	0.907	0.928	0.000	0.000	0.541
	50	0.972	0.999	1.000	0.000	0.000	0.068
alpha = .05	4	0.137	0.145	0.145	0.011	0.005	0.035
	10	0.279	0.461	0.466	0.000	0.000	0.124
	20	0.529	0.815	0.845	0.000	0.000	0.315
	50	0.929	0.998	0.999	0.000	0.000	0.008
alpha = .01	4	0.035	0.034	0.034	0.003	0.001	0.007
	10	0.094	0.192	0.195	0.000	0.000	0.018
	20	0.247	0.545	0.579	0.000	0.000	0.074
	50	0.744	0.975	0.990	0.000	0.000	0.000

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 18:		*half-normal (μ = 0.0, σ = 1.0 for the assoc. normal dist.)*					
alpha = .10	4	0.143	0.146	0.146	0.046	0.035	0.097
	10	0.193	0.262	0.268	0.010	0.026	0.112
	20	0.289	0.440	0.469	0.002	0.014	0.192
	50	0.538	0.753	0.821	0.000	0.005	0.210
alpha = .05	4	0.077	0.074	0.074	0.020	0.015	0.050
	10	0.109	0.154	0.157	0.003	0.011	0.053
	20	0.180	0.294	0.312	0.001	0.005	0.106
	50	0.393	0.623	0.690	0.000	0.002	0.099
alpha = .01	4	0.014	0.015	0.015	0.004	0.002	0.010
	10	0.025	0.040	0.041	0.000	0.002	0.009
	20	0.052	0.099	0.104	0.000	0.001	0.024
	50	0.164	0.331	0.385	0.000	0.000	0.017

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 19:		*0.5·Beta (λ_1 = 1.1, λ_2 = 0.4) + 0.5·Triangular (a = 0.2, M =0.6, b = 0.7)*					
alpha = .10	4	0.801	0.414	0.413	0.000	0.000	0.045
	10	0.999	0.928	0.925	0.000	0.000	0.302
	20	1.000	0.999	1.000	0.000	0.000	0.242
	50	1.000	1.000	1.000	0.000	0.000	0.000
alpha = .05	4	0.664	0.268	0.269	0.000	0.000	0.024
	10	0.995	0.855	0.852	0.000	0.000	0.159
	20	1.000	0.997	0.998	0.000	0.000	0.092
	50	1.000	1.000	1.000	0.000	0.000	0.000
alpha = .01	4	0.311	0.075	0.076	0.000	0.000	0.005
	10	0.913	0.622	0.612	0.000	0.000	0.027
	20	1.000	0.979	0.980	0.000	0.000	0.011
	50	1.000	1.000	1.000	0.000	0.000	0.000

Table 5 – continued

sig. level (right tail)	*sample size*	*Rejection Rates for 20 Alternative Distributions*					
		ks	*cvm*	*sw*	*gini*	*klc*	*MADMASR*
alternative distribution no. 20:		*Bern. (p = .5)·Beta (λ_1 = 1.1, λ_2 = 0.4) + (1 – Bern.)·Triangular (a = 0.2, M =0.6, b = 0.7)*					
alpha = .10	4	0.605	0.219	0.214	0.000	0.000	0.094
	10	0.950	0.637	0.635	0.000	0.000	0.250
	20	1.000	0.944	0.954	0.000	0.000	0.537
	50	1.000	1.000	1.000	0.000	0.000	0.025
alpha = .05	4	0.431	0.129	0.129	0.000	0.000	0.051
	10	0.888	0.490	0.491	0.000	0.000	0.115
	20	0.998	0.890	0.903	0.000	0.000	0.293
	50	1.000	1.000	1.000	0.000	0.000	0.002
alpha = .01	4	0.099	0.047	0.047	0.000	0.000	0.011
	10	0.617	0.235	0.237	0.000	0.000	0.015
	20	0.974	0.713	0.730	0.000	0.000	0.046
	50	1.000	0.998	0.999	0.000	0.000	0.000

Discussion of the Results

The MASMASR Statistic

Critical values.

As Table 1 shows, the critical values increase by a very modest percentage as the upper percentile increases from the 90th to the 99th, with a slight increase in absolute growth as sample size increases. Only upper percentiles are included, because the absolute difference indexed by these values indicates deviation from exponentiality corresponding to a right-tailed alpha. However, nonlinear transformation of the statistic, which may lead to a more reliable test, was not studied here.

The critical values increase more substantially with sample size. A simple, linear relationship is approximated, however, as may be seen by graphing the critical value against the square root of the sample size. Figure 7 shows such a plot, with the square root of n on the x-axis and the 90th percentile of MADMASR on the y-axis. The slope increases almost unnoticeably, proceeding to higher percentiles of the statistic. As the statistic represents variability, the approximately constant ratio of a given percentile to the square root of sample size harbors an at once intuitively satisfying and yet stochastically disappointing relationship. The linear growth of MADMASR without evident convergence indicates that the statistic does not improve in its predictive qualities with increased sample size, as shall be seen in the perspective of Table 2.

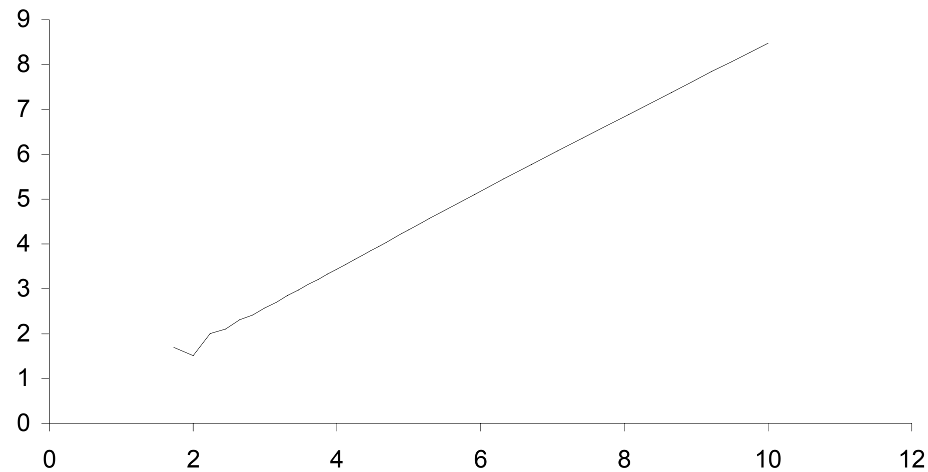

Figure 7. Plot of 90th percentile of MADMASR against square root of n.

Dispersion of MADMASR.

Table 2 shows that the mean value of the researcher's MADMASR statistic continues to grow with sample size, though with decreasing rate of growth, as is illustrated graphically by Figure 8. Further analysis (see Figure 9) of the standard deviation of this statistic, which increases similarly, does not reveal an appearance of conforming to the law of large numbers (Feller, 1957; 1971).

The Parameter Estimate, Theta-Star

Table 2 shows that the estimate bears a normalized trait, with an empirical mean of approximately unity for all values of n. While the MADMASR statistic does not evince convergence characteristics, the standard deviation of

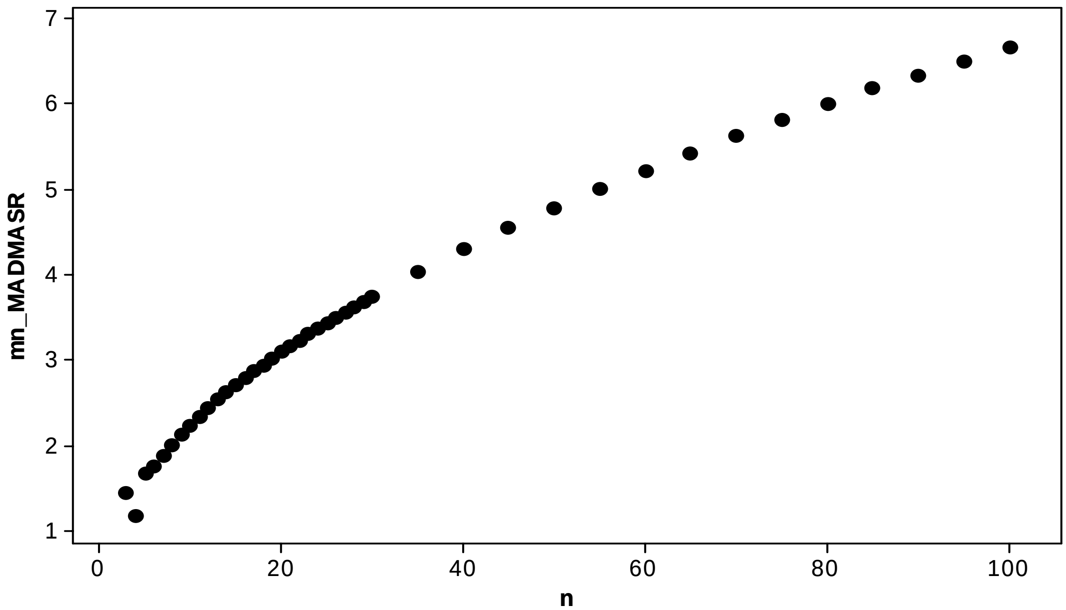

Figure 8. Ratio of the mean of the MADMASR statistic to the sample size.

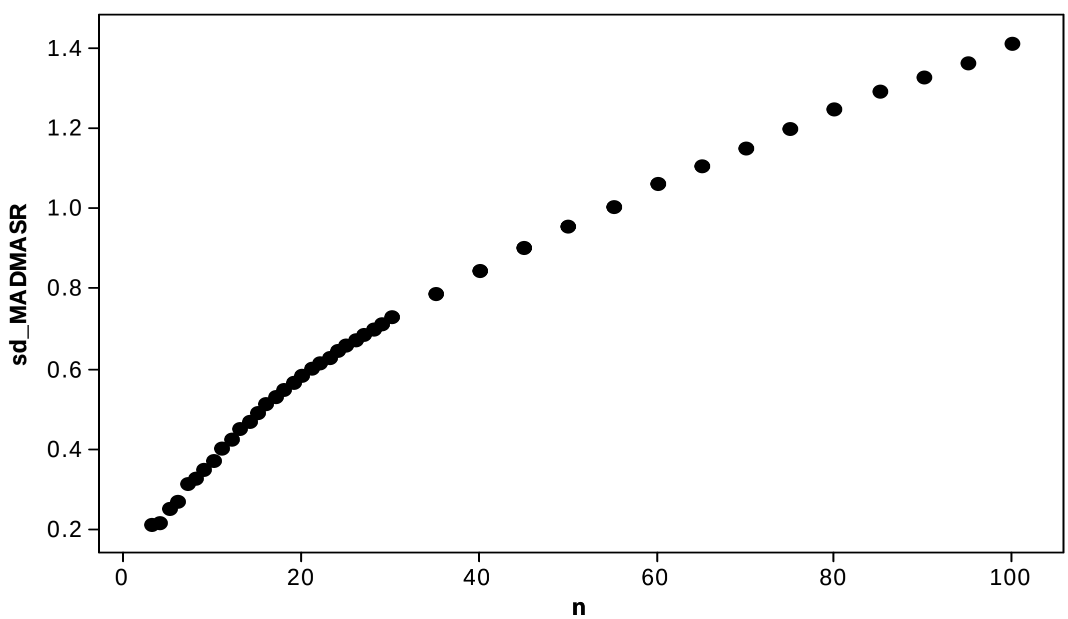

Figure 9. Ratio of the standard deviation of the MADMASR statistic to the sample size.

the estimate decreases with increasing sample size. An empirical analysis suggests adherence to the law of large numbers at least in the weak form, with no evidence contradicting adherence to the strong form (Feller, 1957; 1971). Figure 10 suggests that the ratio of the variance (i.e., the square of the standard deviation in Table 2) of the estimate to the sample size, n, converges to zero.

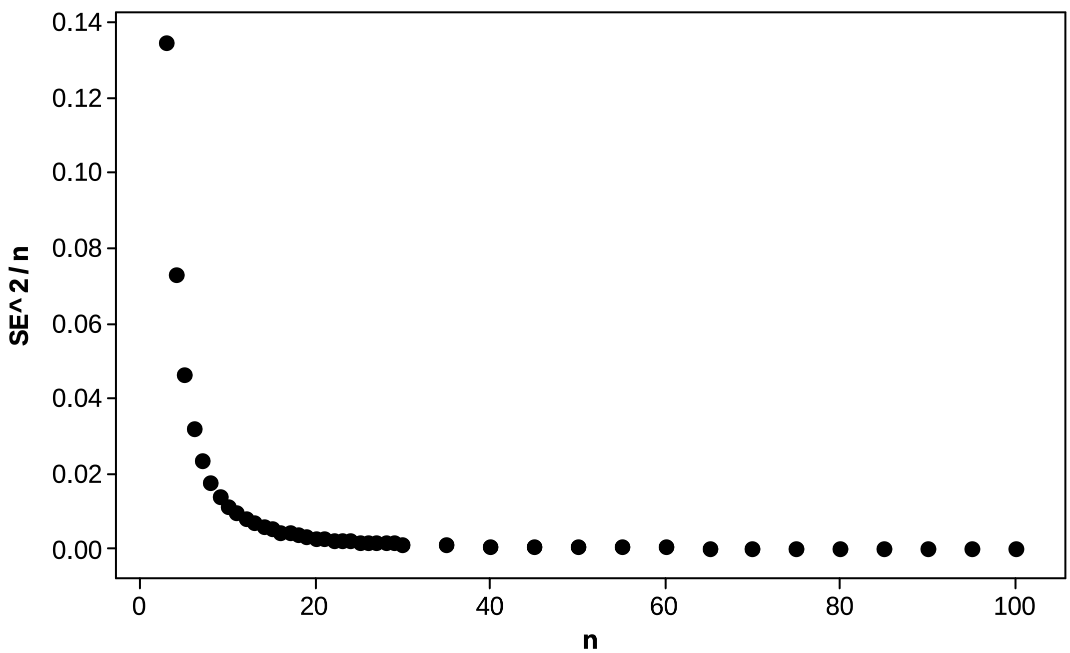

Figure 10. Ratio of the variance of the parameter estimate, Theta-Star, to the sample size.

Insofar as the parameter so estimated is that of an exponential distribution, the mean of which equals its parameter, this relationship of the sampling distribution to sample size is satisfying theoretically, an observation which may be advanced by a regression analysis. Treating the standard deviation of the parameter estimate as a form of standard error, $SE_{theta-star}$, the equation, $SE_{theta-star} = a \cdot n^b$ is conducive to a power regression analysis, utilizing the transformed equation, $\ln\left(SE_{theta-star}\right) = \ln(a) + b \cdot \ln(n)$. Performing linear regression in MINITAB yields Figure 11.

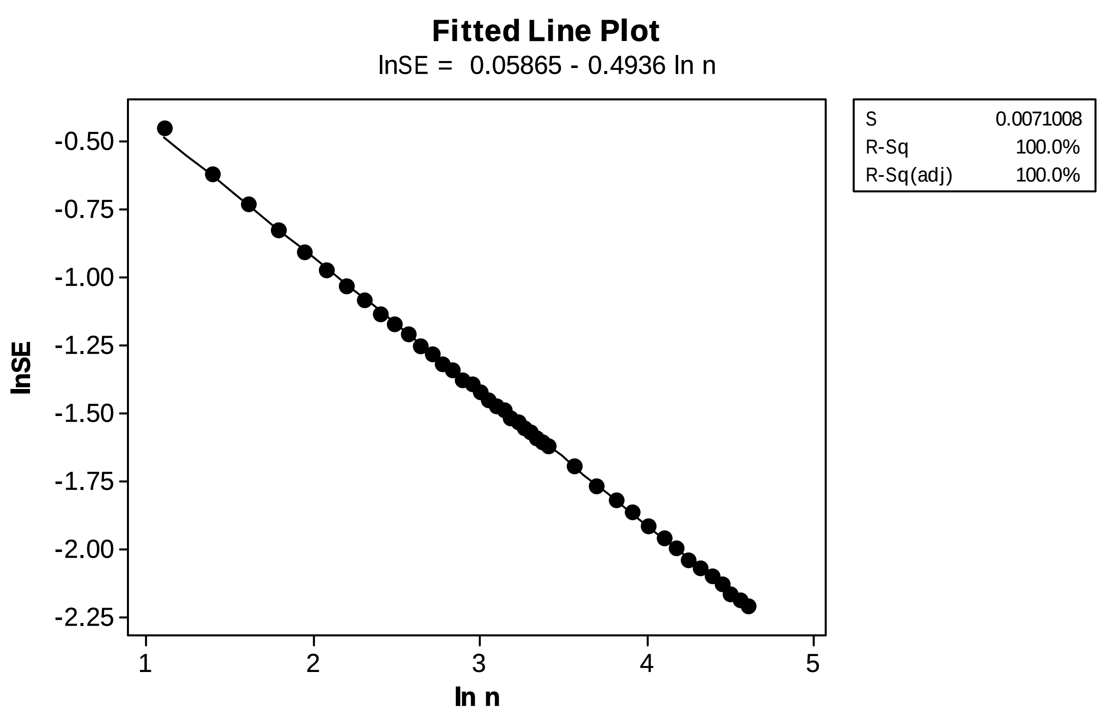

Figure 11. Power regression of the standard deviation of the parameter estimate on sample size (MINITAB).

Inverse transformation of the regression coefficients leads to the equation, $SE_{theta-star} = e^{0.05865} \cdot n^{-0.4936} = 1.06040 \cdot \frac{1}{n^{0.4936}}$. With correlation coefficient very close to unity, this relationship is highly suggestive of the sampling distribution of the sample mean, for which, particularly, $SE_{\overline{x}} = 1.0 \cdot \frac{1}{n^{0.5}}$.

Robustness of MADMASR with respect to Type I Error

Table 3 gives rejection rates for the MADMASR statistic when applied to simulated random samples, n = 4, 10, 20, and 50, with sampling of exponential variates with parameters equal to 0.1, 0.5, 2.0, and 10.0 respectively, for four right-tail alpha values. Hence, under a true null hypothesis of one-parameter exponentiality, these are Type I error rates, which may be compared to the alpha values employed, .10, .05, .025, and .01 respectively. Additionally, the average theta-star parameter estimates and standard deviations are given, for comparison with the theoretical values for the variates sampled. The rejection rates and average estimates are for 100,000 replications in each case.

The estimated parameters are within two percent of the theoretical values for the three largest sample sizes in all cases and no more than 3.2 percent off, in the cases with n = 4. Coefficients of Variation (CV) may be observed, dividing the standard deviations by the average estimates. Consistent with the pattern discussed above with respect to table two, the CV decreases with increased sample size and holds similar values for all four exponential distributions. For n =

4, the CV ranges from 52.3 to 52.7 percent; for n = 10, from 33.6 to 33.8 percent, for n = 20, from 24.0 to 24.1 percent, and for n = 50, from 15.0 to 15.4 percent.

The rejection rates for 100,000 replications are consistent with the alpha levels representing the critical values calculated in the initial Monte Carlo sampling procedure. As may be seen by viewing Table 3, the discrepancies relative to the true, empirical alpha values are predominantly within 1 or 2 percent and less frequently as large as 4 or 5 percent. The methodology behind Table 3 performs a validating role for the critical values initially stored, a role which has been reasonably fulfilled for the purposes here, supporting a conclusion of robustness with respect to Type I error.

It should be noted that additional support for robustness of the researcher's test with respect to Type I error is evidenced in Table 4, for alternative distribution number 14, a Chi-Square distribution with 2 degrees of freedom, equivalent to an exponential distribution with parameter equal to 2.0. This portion of Table 4 demonstrates rejection rates consistent with the empirical upper-tail areas, with tolerances similar to those seen in Table 3.

Robustness of MADMASR with respect to Type II Error

The notion of robustness applies in different contexts or dimensions of varying effects and may be more or less easily focused, depending on how succinctly the context can be described. Thus, a straightforward definition holding a statistical procedure to be robust "with respect to a particular postulated assumption, if the procedure is relatively insensitive to (slight) departures from the assumption” (Hollander & Wolfe, 1973, p. 460) may adequately illuminate the issue in robustness of Type I error rate with respect to violation of the assumption of normality in an independent samples t-test, and yet variations depending upon significance level and type of departure, variances, and sample sizes, as well as interrelations between these and other conditions would persist (see Bradley, 1986, pp. 26-28). Bradley gives examples of several factors which may cause distortion of Type I or Type II error, cautioning that his list is “illustrative rather than exhaustive”, and that “the factors interact with whatever violation occurs” (1968, pp. 27, 26 respectively). Moreover, paradoxes are observed, reflecting the elusive nature of robustness, indeed, the frequent unpredictability of how robustness may be affected by combinations of factors (Bradley, 1968, p. 28). Bradley, whose work on robustness is seminal, observes the lack of agreement either on a definition of robustness or on commonly accepted criteria for evaluating robustness (1968, p. 26). Van Belle, Fisher, Heagerty, and Lumley compare robustness to beauty:

> Depending on the specific criteria for beauty, there may be greater or lesser agreement about the beauty of an object. Similarly, different statisticians may disagree about the robustness of a particular statistical procedure depending on the probability distributions of concern and use of the procedure. Nevertheless ... the concept ... proves to be useful conceptually and in discussing the range of applicability of statistical procedures (2004, p. 254).

In respect to Type II error, where, in place of a concise null condition there may be an unspecified range of alternatives in addition to any underlying assumptions, a multidimensional complex of varied conditions afflicts the effort to circumscribe the robustness of a statistical procedure such as goodness-of-fit testing. With multiplied force, Bradley's caveats concerning interacting effects and the want of established benchmarks position the problem between highly subjective assessment and the inclination to partition the issue into more workable contexts described by narrowed ranges of conditions.

Monte Carlo methods – and modern computers – attenuate the subjectivity with which robustness has been regarded, although there exist a wide variety of approaches to robustness (Hampel, Ronchetti, Rousseeuw & Stahel, 1986, p. 4, passim). In the context of estimation, robustness can be seen as "sensitivity ... to small changes in the underlying distribution" (Lehmann, 1998, p. 190; and see Hampel, 1986). With respect to Type I error, the actual probability of rejecting the null hypothesis differs from the nominal significance level because the underlying probability distribution differs from that which was used to derive the criteria for the nominal level (see van Belle et al., 2004, p. 254). In some contexts, gauging sensitivity to small changes can be operationalized by employing contaminated distributions, in which one density function predominates but is mixed with a small proportion of variates following a different, contaminating density, "because the interest is in how well a standard procedure holds up when the data are subject to contamination by a different population or by incorrect measurements" (Gentle, 2003, p. 169). The mixture distributions, particularly AD-01 and AD-02, in the current study inhabit a distinct context, because rather than both Type I and Type II error being framed by the same assumption regarding the underlying distribution as in the case of a t-test, here the alternative hypothesis itself embodies unspecified deviation from the null exponential distribution.

With the foregoing background, a focus on the present issue may be framed in terms of stability of rejection rates for the MADMASR procedure over a range of alternative distributions. In so doing, an attempt will be made to transcend the limited perspective in much applied research, such as reliability or survival studies. An example of that perspective is the study of the robustness of testing for exponentiality with respect to Weibull alternatives (Zelen & Dannemiller, 1961; and see Spurrier, 1984, pp. 1649-1650). In omnibus goodness-of-fit testing, the more focused perspective persists, as may be seen in the sequences of alternatives from the same distribution family which make up the set of alternative distributions in the typical goodness-of-fit study, as in the present one.

The larger perspective can be sought in Table 4, restricted to the .05 significance level and, to the extent further demarcation is indicated, to the lowest sample sizes. In this light, a view emerges for certain of the alternative distributions in which stability of rejection rates may be seen, across a variety of effects or treatments, in Table VI. For comparison, AD-14 has been included in Table VI, for the reason that the rejection rates achieved are fairly close for this exponential distribution, as well as for the gamma, half-normal, Weibull, Gumbel, lognormal, Makeham, and Chi-square alternatives shown, the latter comprising 11 of the 19 non-exponential distributions in the power study for the MADMASR statistic. While there is a degree of stability in power for at least the lower sample sizes, which may be interpreted as robustness with respect to Type II error rate, the clustering of these non-exponential rejection rates alongside those for AD-14 around the nominal significance level of .05 does not augur well for MADMASR. For these distributions, the procedure may attain, in an apparent modicum of Type II robustness, the paradoxical status of a test which is reliable while of questionable validity.

Overall, restricting sample size to n = 10 and alpha to the .05 level, 14 among the 19 non-exponential alternatives, to wit, AD-03, AD-04, AD-06, AD-07, AD-08, AD-09, AD-10, AD-12, AD-15, AD-16, AD-17, AD-18, AD-19, and AD-20, have rejection rates strictly between .04 and .16 (the lowest being .0420 for AD-10 and the highest being .1588 for AD-19, with .1242 and .1149 respectively for AD-17 and AD-20). This range of .1168 or less than a 12 percent spread for 14, or approximately 74 percent, of the alternatives studied provides an additional index of Type II error rate robustness. These indications of stability for the MADMASR statistic will suffice to proceed to the matter of power.

Table VI

Stability of Rejection Rates for Selected Alternatives and Three Lowest Sample Sizes, at the Alpha = .05 Significance Level.

sample size	AD-03 gamma *($\sigma = 0.625$, $\lambda = 1.6$)*	AD-04 gamma *($\sigma = 0.25$, $\lambda = 4.0$)*	AD-18 *half-normal ($\mu = 0.0$, $\sigma = 1.0$ for the assoc. normal)*
4	0.0538	0.0531	0.0501
10	0.0542	0.0516	0.0527
20	0.0659	0.0720	0.1064

	AD-06*Weibull* *($\sigma = 1.10773$, $\lambda = 1.5$)*	AD-07 *Weibull* *($\sigma = 1.128387$, $\lambda = 2.0$)*	AD-08 *Gumbel* *($\mu = 0.42278$, $\sigma = 1.0$)*
4	0.0532	0.0499	0.0535
10	0.0537	0.0533	0.0515
20	0.0866	0.0897	0.0575

	AD-09 *lognormal (x; $\mu = 0.0$, $\sigma = 0.2$) where μ = mean(ln(x)) and σ = sd(ln(x))*	AD-10 *lognormal ($\mu = 0.0$, $\sigma = 1.0$)*	AD-12 *Makeham ($\xi = 1.0$, $\lambda = 1.0$, $\theta = 1.0$) with DLMF parameterization*
4	0.0492	0.0486	0.0512
10	0.0517	0.0420	0.0522
20	0.0617	0.0214	0.0919

	AD-14 *chi-square, df = 2 [exp($\theta = 2.0$)]*	AD-15 *chi-square, df = 3*	AD-16 *chi-square, df = 4*
4	0.0509	0.0525	0.0549
10	0.0503	0.0549	0.0551
20	0.0493	0.0660	0.0725

Power of MADMASR: Rejection Rates For A False Null Hypothesis

The relationship of Table 4 to Table 5.

Table 4 focuses on rejection rates for the researcher's MADMASR statistic on testing samples from alternatives to the exponential distribution. Four decimal place rejection rates are given for four right-tailed alpha values. For three of the alpha values, excluding .025, the rejection rates are repeated in Table 5, which gives three decimal place values, and which compares five other goodness-of-fit test statistics with that of the researcher.

As treated above, Table 4, being the first presentation of the alternative distributions, contains average means and variances for the simulated variates which were used to verify consistency of the simulations with theoretical values.

Focusing on MADMASR: Table 4.

One feature which stands out in at least 13 of the 19 non-exponential alternatives is that power of the MADMASR test decreases dramatically as sample size increases to n = 50. (MADMASR is similar to the klc statistic in this respect, of the tests observed here.) It is suspected that this is a result of an accommodation of deviations in the processing of absolute deviations which works against the effectiveness of the researcher's goodness-of-fit procedure. In recasting the study, it may be profitable to focus on a range of smaller sample sizes in an attempt to locate the range in which the MADMASR procedure may be most effective. The exceptions are: alternative distribution number 3 (AD-03), in which there is increased power for the .10 and .05 columns (thus the phenomenon differentiates among gamma distributions); AD-06, in which a smaller decrease occurs, while power for n = 50 remains larger than or comparable with that for n = 4 and n = 10 (thus the phenomenon differentiates among Weibull distributions); AD-12 (Makeham), where power increases with sample size for all columns; AD-15, where power increases with sample size for .10 and perhaps .05 and remains higher for n = 50 than for the two smallest sample sizes for .025 and .01; AD-16, where a smaller decrease occurs for n = 50, without dropping below power for n = 4 and n = 10 for the first two alpha values (thus the phenomenon differentiates among Chi-Square distributions); and AD-18 (half-normal), where power increases for the .10 column and remains above power for n = 4 and n = 10 for the remaining three columns. (In the case of AD-14, where power remains stable for all sample sizes, the rejection rates actually reflect Type I error.) In four of the 13 distributions in which the feature shows up, AD-02 (mixed exponential and normal), AD-05 (Weibull with 0.5 parameters), AD-11 (lognormal, σ = 2.0), and AD-19 (first anomalous), power is near or indistinguishable from zero. In at least the latter case, further investigation may be warranted to determine whether an algorithmic artifact is being observed, or power in fact vanishes for larger samples.

It may further be seen from Table 4 that the researcher's goodness-of-fit procedure is optimal for different ranges of sample sizes, depending on the alternative distribution, and that optimality exists for each of the four sample sizes, for different groups of distributions. While the researcher has not categorized the alternative distributions according to available criteria, such as those presented in the literature in reliability analysis (see Chapter 2), further investigation in this regard might be worth pursuing.

For alpha = .10, power of .2 or higher is achieved, at least at one sample size, for AD-12 (Makeham, n = 50, power = .2271), AD-17 (uniform, n = 10, power = .2524 and n = 20, power = .5412), AD-18 (half-normal, n = 50, power = .2099, with power of .1924 for n = 20), and AD-19 and AD-20 (the two anomalously-shaped distributions). For AD-19, power is .3016 and .2418, respectively, for n = 10 and n = 20. For AD-20, power is .2501 and .5370, respectively, for n = 10 and n = 20. The relatively high powers achieved for the last two distributions, being simulated finite populations intended to evoke “messy” data which may be found empirically in the sentient world, may indicate redeeming value for the MADMASR goodness-of-fit procedure. These two distributions are highly selective, and a more general study of such simulated mixtures would be of interest. Power remains relatively high (for MADMASR) for alpha = .05, for AD-17 (n = 20, power = .3151) and for AD-20 (n = 20, power = .2932).

Comparing The Power Of Goodness-Of-Fit Tests

Abbreviations.

In this discussion, the abbreviations used in Table 5 for the various goodness-of-fit tests will be applied. These are as follows:

KS................. Lilliefors-Kolmogorov-Smirnov statistic (Lilliefors, 1969)
CVM..............Cramer-von Mises statistic with K transformation (Seshadri, Csorgo, & Stephens, 1969)
SW.................Shapiro-WIlk statistic (Metz, Haccou, & Meelis, 1994)
Gini................Gini statistic (Gail & Gastwirth, 1978)
KLC.................Kullback-Liebler information statistic, with Correa's entropy estimator (Choi, Kim, & Song, 2004)
MADMASR...Maximum Absolute Deviation from the Median Absolute Standardized Residual

The statistics will be considered in the order, given above, in which they appear from left to right in Table 5.

Power: Overview of performance of the five alternative tests.

The KS statistic varies in power over the 20 distributions from below .2 in a few cases to very high (near unity), sometimes registering at the upper extreme, in some cases at smaller sample sizes or smaller alpha levels or both. Generally, this statistic performs well, at least as well if not better than the others, although there are exceptions. Except for the results for AD-14 (Chi-Square reduced to exponentiality), which do not represent power, the KS consistently performs with power that increases substantially as the sample size increases, except where the limit of unity has been achieved at a lower sample size.

The CVM statistic is similar in performance to the KS statistic, typically performing at somewhat lower powers than the latter, especially at smaller sample sizes, although in several cases it outperforms the latter statistic. As does KS, CVM increases in power for larger sample sizes. The CVM statistic consistently outperforms KS for AD-12 (Makeham), AD-17 (uniform), and AD-18 (half-normal).

The SW statistic varies with comparatively worse performance for some distributions, while of comparable power in some cases to CVM and even to KS. SW performs better than or almost as well as KS for the Weibull distributions (AD-06 and AD-07) , Makeham (AD-12), and for Chi-Square, df = 3 (AD-15), comparable to CVM for Chi-Square, df = 4 (AD-16), extremely well (the best) for uniform and half-normal (AD-17 and AD-18), and comparable to CVM for the anomalously-shaped mixtures (AD-19 and AD-20), for which KS has considerably higher power at n = 4. The power of SW decreases with increasing sample size for the mixed exponential (AD-01), the first Weibull (AD-05), the second and third lognormals (AD-10 and AD-11), and for Chi-Square, df = 1 (AD-13),

The Gini statistic performs with the highest power of the tests here for four of the distributions, AD-01 (mixed exponential), AD-05 (first Weibull), AD-11 (third lognormal), and AD-13 (Chi-Square, df =1), and second only to KS for AD-08 (Gumbel). For some distributions, the Gini statistic shows a decrease in power with increasing sample size. This occurs for AD-02 (mixed exponential and normal), AD-03 (first gamma), for AD-06 and AD-07 (the second and third Weibull, for which power vanishes for most cases), for AD-12 (Makeham), AD-15 and AD-16 (Chi-Square, df = 3 and df = 4, with vanishing power), and for AD-17 and AD-18 (uniform and half-normal, with vanishing power). For AD-04 (second gamma) and for AD-19 and AD-20, the anomalously-shaped mixtures, power vanishes entirely for the Gini statistic.

Except for the Gumbel distribution which presents computational difficulties for the KLC statistic and is not represented in the column for that statistic, the KLC and Gini statistics perform similarly in regard to power at lower sample sizes. However, KLC harbors an important distinction from Gini, as well as from the other tests except for MADMASR: more consistently than it does for the latter, power decreases for the KLC statistic with increasing sample size. Thus, KLC performs the highest for AD-01 (the mixed exponential) but only at n = 4. KLC performs second to Gini for AD-05, AD-11, and AD-13 (the first Weibull, the third lognormal, and Chi-Square, df = 1), but only at n = 4 in each case. In the case of AD-10 (the second lognormal), KLC comes in second to KS, but only at n = 4.

Schematic view of comparative power over the alternative distributions.

For AD-01, KS is superior to CVM, SW, and MADMASR, but its power is exceeded by Gini, consistently but especially at lower sample sizes. Its power is only exceeded by KLC for n = 4, in which case alone, KLC is the highest power performer. For the other sample sizes, Gini is decidedly the highest power performer. The SW and MADMASR tests do not furnish power as large as .1 in any case here.

A summary of comparative performance is provided in Table VII for AD-01 and the remaining distributions, choosing the .10 level of significance because the focus of this study is the researcher’s MADMASR statistic for which power is generally low, and choosing n = 4 and n = 20 to avoid the decreased power of MADMASR at n = 50, presenting instead very small and medium sample sizes.

For AD-02, KS is decidedly the highest powered test at alpha = .10, although KS is outperformed by both CVM and SW at lower significance levels and the smallest sample size here. While KLC exceeds .1 in power for the smallest sample size and larger alphas, and Gini performs slightly better, MADMASR rises above .1, to .161 in only one case, for n = 10 at alpha = .10.

For AD-03, while KS is strictly higher in power, the first three statistics, KS, CVM, and SW are roughly comparable in power for all cases. The power for Gini and KLC is negligible, well under .1 in all cases and these two tests are outperformed in all cases here by MADMASR. The latter achieves power, increasing with sample size, which consistently exceeds .1 for alpha = .10 (yielding .105, .108, .120, and .136 respectively) and consistently exceeds .05 for alpha = .05.

For AD-04, KS strongly outperforms all other statistics. CVM and SW perform almost alike, sharing second position in power. Gini and KLC both show zero power to three decimal places, while MADMASR performs with less consistency than, but otherwise similar to, its rejection rates for the previous distribution, also gamma, exceeding .1 in power for the three lower sample sizes at alpha = .10, yielding .100, .108, and .143 respectively. In the case of this distribution, the power of MADMASR generally rises, then drops drastically for n = 50.

For AD-05, KS performs consistently well but is decidedly outperformed by the Gini statistic at all levels and, for n = 4, is outperformed by the KLC statistic as well. CVM performs adequately only at the higher two sample sizes. SW and MADMASR show negligible power, well below .1 in all cases. Anomalously, the SW statistic decreases in power with increasing sample size.

For AD-06, KS, CVM, and SW are the best performers, KS outperforming the others at the lower sample sizes, while at the higher sample sizes, SW has highest power, CVM is slightly lower, and KS is lower by a greater, though not substantial margin. Gini and KLC have negligible power for this distribution. The performance of MADMASR in the case of this second Weibull distribution resembles its performance generally in respect to the gamma distributions AD-03 and AD-04, and the next, third Weibull, AD-07, although not the first Weibull, AD-05. The power of MADMASR at alpha = .10 exceeds .1, with .102 and .111 at the first two sample sizes, increasing to .165 for n = 20 and falling to .146 for n = 50.

For AD-07, the KS statistic outperforms all others, with SW and CVM trailing (more so at lower sample sizes and at lower alpha values) as second and third in power. Gini and KLC vanish in power for almost all cases. MADMASR has power close to but below .1 for two cases, .094 at n = 4 for alpha = .10 and .090 for n = 20 at alpha = .05, and exceeds .1 for the middle two sample sizes at alpha = .10, .113 at n = 10 increasing to .174 for n = 20, and falling to .064 for n = 50.

For AD-08, the two best performers are KS and Gini respectively. CVM and SW perform similarly, with CVM slightly higher in power in most but not all cases. The power of the latter two statistics is substantially less than that of KS and Gini, with a smaller difference for the higher sample sizes. KLC is omitted for this distribution (Gumbel) for computational reasons. The performance of MADMASR here resembles its performance for the gamma and second two Weibull distributions, with power rising from .102 at n = 4 to .106 at n = 10 and to .119 at n = 20 and then falling to .045 at n = 50 for alpha = .10. The resemblance remains for lower alpha values, though power is well below .1, as in respect to the other distributions.

For AD-09 (lognormal with $\sigma = 0.2$), KS performs extremely well, with power registering as unity for the three larger sample sizes at all levels. This is not the case for the second lognormal alternative, AD-10 (with $\sigma = 1.0$), which may be explained by a density curve for AD-09 which is more symmetrical in shape, indeed, near bell-shaped for shape parameters as close to zero as 0.2, and thus clearly differentiated from an exponential density curve, as opposed to those close to or above unity (see Figure 12). CVM and SW perform similarly with SW very slightly higher in power in some cases and with decent performance at the higher sample sizes. At the alpha = .10 level, MADMASR rises from .092 at n = 4 to .108 at n = 10 and to .131 at n = 20, then falling to .022 at n = 50. Patterns are similar at higher alpha values, but with substantially lower power. Power for Gini and KLC vanishes at all levels for this alternative.

For AD-10, none of the tests achieve adequate power, possibly excepting .381 for KS at alpha = .10 and n = 50. KS is the best performing statistic of all here, with similar values for the other five statistics at n = 4, but divergence for higher sample sizes, because SW, KLC, and MADMASR decrease in power with increasing sample size. CVM and Gini perform similarly, with the former being somewhat superior at the highest sample size, n = 50. SW does not achieve power of .1 in any of the cases shown, while MADMASR comes close (.098 and .085) only for the two smallest sample sizes, at alpha = .10 .

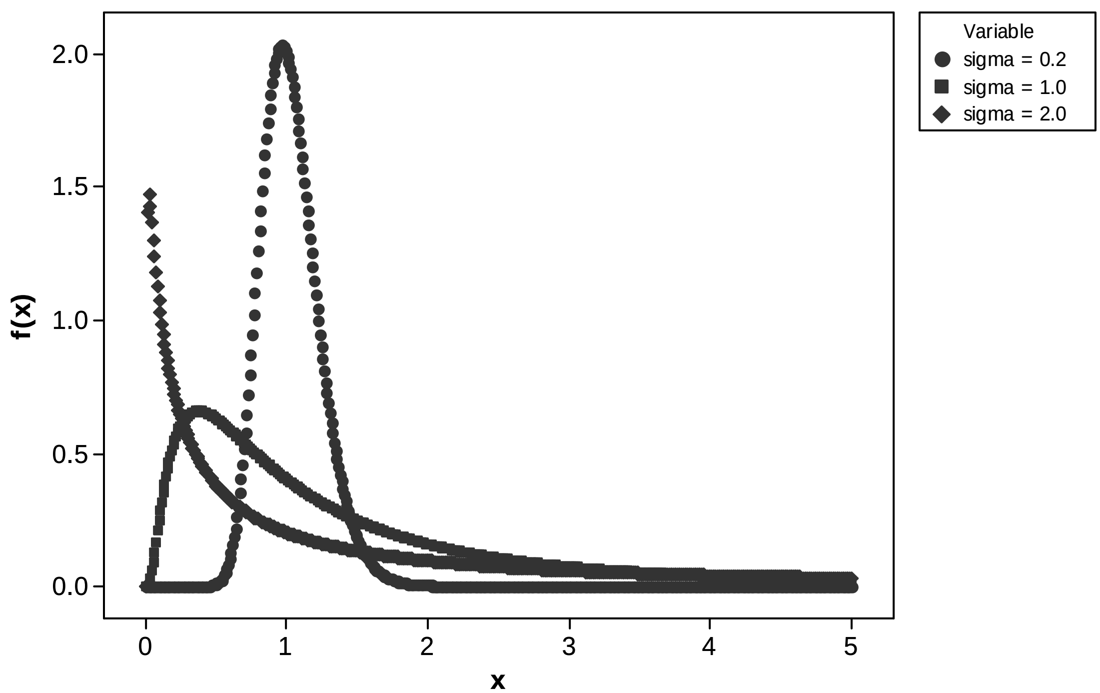

Figure 12: Lognormal density curves with parameter mu = 0.0.

For AD-11 (lognormal with $\sigma = 2.0$), while the density is a decreasing function, like the exponential, this lognormal decreases with much more abruptly, having an asymptote at the vertical axis rather than an intercept, so that AD-11 is differentiated in the opposite way from AD-09 (see Figure 12). This may result in higher power than in the case of AD-10 for some of the tests, particularly for KS and CVM which perform consistently across these distributions. Gini, which does not perform well for the other lognormals, is the best performer here, followed by KS. CVM also performs well for the two higher sample sizes, while KLC outperforms all of the other distributions at n = 4 and the lower alpha values and is a close second at alpha = .10. KLC and MADMASR do not provide adequate power for the higher three sample sizes (though KLC achieves .131 at n = 10, alpha = .10). SW remains far below .1 in power for all cases.

For AD-12 (Makeham), four tests achieve power over .1, while Gini and KLC , both with power decreasing with increasing sample size, remain below .1 in power, decreasing from .067 and .058 respectively. The best performers, CVM and SW, are very close at lower sample sizes, with SW performing the best at the higher sample sizes here. The KS statistic trails CVM and SW, especially at higher sample sizes. While the level of power achieved by any test here is less than may be desired, MADMASR – which increases in power with increasing sample size – performs respectably in comparison here, in close fourth place after KS, even exceeding the power of KS for n = 20.

For AD-13, Gini is the best performer, followed by KS. At n = 4 only, KLC outperforms KS and comes close to Gini, outperforming it too at alpha = .01, but decreases in power with increasing sample size. Opposite to KLC, CVM remains negligible in power for the lower sample sizes but increases substantially in power for the higher sample sizes. While yielding higher power than SW for this alternative distribution, MADMASR remains below .1 in power.

AD-14 reduces to an exponential pdf, so that the rejection rates reflect Type I error rather than power.

For AD-15, Gini and KLC are negligible in power. KS, SW and CVM perform best, with CVM being dominant at higher sample sizes and at lower values of alpha. The three latter statistics achieve similar levels of power overall. MADMASR, which increases in power with increasing sample size here, performs better than Gini and KLC (which decrease in power with increasing sample size for this distribution) but exceeds .1 in power only for alpha = .10.

For AD-16 (df = 4 vs. df = 3 for AD-15), as should be expected for Chi-Square with higher degrees of freedom, greater power is achieved in distinguishing the density function from an exponential pdf, for those tests performing effectively. Chi-Square with df = 2 reduces to an exponential pdf, which is a strictly decreasing function, while Chi-Square pdf's with higher degrees of freedom have progressively wider intervals above zero on which they are strictly increasing, and tend progressively towards approximate symmetry (and normality, at much higher degrees of freedom). As with the previous alternative, Gini and KLC have negligible power. For AD-16, KS is in clear lead for power, followed by CVM and SW which lie close together, except at n = 50 where CVM prevails. MADMASR exceeds .1 for alpha = .10, increasing to a high of .140 at n = 20.

For AD-17 (unit uniform), CVM and SW perform very well, closely placed at the two smaller sample sizes, with SW somewhat higher in power for the larger sample sizes. KS comes in third, trailing substantially at the two larger sample sizes. Gini and KLC have negligible power. MADMASR performs very well, relative not only to its performance elsewhere but to the rejection rates overall in the power study, for mid-sized samples at alpha =.10, with power of .252 at n = 10 and .541 at n = 20.

For AD-18 (half-normal), SW and CVM perform best, with SW leading by an amount that increases with sample size. KS trails in third place. Gini and KLC achieve negligible power. At alpha = .10, MADMASR increases from .097 at n = 4 to .210 at n = 50, and also exceeds .1 in power at alpha = .05, with .106 at n = 20 and .099 at n = 50.

For AD-19, KS achieves very high power. CVM and SW trail behind KS at n = 4 and at n = 10 for alpha = .01, but perform nearly as well as KS otherwise, with little difference between the two tests. Gini and KLC vanish in power, given to three decimal places, for all cases here. MADMASR achieves powers in excess of .1, to wit, .302 and .242, respectively, at n = 10 and n = 20 for alpha = .10 and .159 for n = 10 at alpha = .05.

For AD-20, the chief performers are similarly arrayed as for the previous distribution, but with less spectacular power for KS, CVM, and SW, and higher power for MADMASR. KS achieves high to very high power, but falls just below .1 at n = 4 and alpha =.01. CVM and SW lie close together, with SW slightly ahead most of the time, falling substantially behind KS as sample size decreases. Gini and KLC vanish in power for all cases. At alpha = .10, MADMASR rises from .094 to .250 and .537 at n = 10 and n = 20, then falls well below .1 for n = 50 (.025). MADMASR also exceeds .1 in power at alpha = .05, with .115 at n = 10 and .293 at n = 20. Anomalously, though consistent with the general pattern for this test, MADMASR has negligible power at n = 50, in spite of its high power elsewhere for this alternative distribution relative to its performance on other alternatives.

Summary of performance of the alternative tests.

In 12 of the 19 alternative distributions proper, the KS statistic appears at least once as the best performer in power, for the two cases per distribution summarized in Table VII. CVM shows up first for only 3 of the 19 alternatives, yet does show up among the three best for each of the 19 alternatives. SW shows up at least once as best for 5 of the 19 alternatives and shows up among the best four in 14 of the 19 alternatives. The Gini statistic appears in first place at least once in 4 of the 19 alternatives and shows up among the best five performers in 7 of the 19. The KLC statistic shows up as first in 1 of the 19 alternative distributions and shows up among the best four in 6 of the 19 alternatives. It should be noted that only cases where power is close to .1 (above .090) are shown in Table VII, apart from the right-most column for the best performance of MADMASR.

Summary of the performance of MADMASR.

MADMASR never shows up as first or second best. It shows up in 1 case, for AD-10, as third best, with power of .098. It shows up as the fourth best in 12 of the 19 alternatives, ranging in power from near .1 to highs of .541 and .537 for AD-17 and AD-20 respectively. In addition, MADMASR shows up as fifth best (and over .1) for 1 alternative, AD-08. Thus, MADMASR shows up among the five best performers near or over .1 in power in 14 of the 19 alternatives. In 5 of the 19 alternatives, MADMASR evinces power exceeding .2, to wit: .227 for AD-12 (Makeham); .541 for AD-17 (uniform); .210 for AD-18 (half-normal); .302 for AD-19 (linear combination of Beta and triangular), and .537 for AD-20 (mixture of complementary Bernoulli products of Beta and triangular).

The MADMASR statistic shows potential for some distinctive distributions. While reaching only .136 and .140 as maximal powers for the Chi-Square distributions with df = 3 and df = 4 (AD-15 and AD-16, respectively), it is likely that MADMASR would perform progressively better for Chi-Square with higher degrees of freedom and for normal distributions, although pure examples of the latter were not tested in this study. MADMASR did not perform well for the mixture of exponentials (AD-01), but did achieve a maximum of .161 for AD-02, a mixture of exponential and normal and performed much better for the anomalous mixture populations (AD-19 and AD-20). Mixtures of exponential and normal distributions with different parameters and other mixture distributions appear to present fertile ground for exploration of MADMASR. In a study of the present character, relatively few alternatives of the broad variety available can be tested and representatives of a single distribution family are highly selective. In a more focused study, specific types of mixtures could be studied methodically, to explore the presence of patterns in performance related to distributional characteristics, parametric or otherwise.

Aside from mixture distributions, MADMASR performed exceptionally well for the Makeham distribution (AD-12) and showed potential for the Weibull distributions (AD-06 and AD-07), with optimal powers of .165 and .174 respectively. These distributions have been useful in survival or reliability analysis, in the biological and industrial contexts, respectively. As in the case of mixtures, a more focused study could methodically investigate the performance of a specific test or tests, in respect to a range of distributions of one specific family or type. The remaining two distributions for which MADMASR performed well were the uniform and half-normal distributions, each unique, save for linear translation or scaling on the horizontal axis. The uniform distribution (AD-17), for which MADMASR achieved power of .541 for $n = 20$ at alpha = .10, plays a fundamental role in goodness-of-fit and in the study of probability distributions generally. The half-normal density (AD-18) is a decreasing function with an inflection point at change of concavity, unlike the simple exponential distribution.

The latter density function has a strictly positive second derivative indicating that its graph is concave up on its entire interval of support. It is of interest that MADMASR showed one of its better performances (.192 at n = 20, alpha = .10) for such a distribution, as the presence of an inflection point is a principal distinguishing feature among monotonic functions.

Table VII

Summary of Comparative Performance of Goodness-of-fit Tests

Alt. Dist.	n	highest power at $\alpha = .10$ and n for this alt.	Other tests with power close to or above .1 at $\alpha = .10$ for sample sizes 4 and 20				highest power of MADMASR for this alt. distribution
AD-01	4	klc .307	gini .301	ks .163			.076 n = 4,
	20	gini .830	ks .663	cvm .367			α =.10
AD-02	4	ks .308	cvm .275	sw .275	klc .203	gini .201	.161 n = 10,
	20	ks .982	cvm .963	sw .305			α =.10
AD-03	4	ks .152	cvm .128	sw .127	mad .105		.136 n = 50,
	20	ks .331	sw .328	cvm .323	mad .130		α =.10
AD-04	4	ks .422	cvm .173	sw .173	mad .100		.143 n = 20,
	20	ks .987	sw .752	cvm .751	mad .143		α =.10
AD-05	4	gini .476	klc .451	ks .314			.061 n = 4,
	20	gini .971	ks .913	cvm .736			α =.10
AD-06	4	ks .194	cvm .154	sw .154	mad .102		.165 n = 20,
	20	sw .580	cvm. 563	ks .546	mad .165		α =.10
AD-07	4	ks .346	sw .193	cvm .192	mad .094		.174 n = 20,
	20	ks .928	sw .863	cvm .845	mad .174		α =.10
AD-08	4	ks .578	gini .525	cvm .175	sw .175	mad .102	.119 n = 20,
	20	ks .976	gini .779	cvm .774	sw .759	mad .119	α =.10
AD-09	4	ks .990	cvm .208	sw .208	mad .092		.131 n = 20,
	20	ks 1.000	sw .904	cvm .900	mad .131		α =.10
AD-10	4	ks .127	klc .110	mad .098			.098 n = 4,
	20	ks .224	gini .169	cvm .120			α =.10

Table VII – continued

Alt. Dist.	n	highest power at $\alpha = .10$ and n for this alt.	Other tests with power close to or above .1 at $\alpha = .10$ for sample sizes 4 and 20			highest power of MADMASR for this alt. distribution
AD-11	4	gini .473	klc .465	ks .311		.059 n = 4,
	20	gini .975	ks .928	cvm .833		α =.10
AD-12	4	cvm.124	sw .124	ks .115	mad .099	.227 n = 50,
	20	sw .280	cvm .258	ks .157	mad .168	α =.10
AD-13	4	gini .320	klc .300	ks .186		.080 n = 4,
	20	gini .763	ks .583	cvm .250	klc .136	α =.10
AD-14	4	N.A.				
	20	N.A.				
AD-15	4	ks .139	cvm .124	sw .124	mad .103	.136 n = 50,
	20	sw .293	cvm .287	ks .275	mad .127	α =.10
AD-16	4	ks .196	sw .140	cvm .139	mad .105	.140 n = 20,
	20	ks .562	sw .458	cvm .457	mad .140	α =.10
AD-17	4	cvm .253	sw .253	ks .227		.541 n = 20,
	20	sw .928	cvm .907	ks .682	mad .541	α =.10
AD-18	4	cvm .146	sw .146	ks .143	mad .097	.210 n = 50,
	20	sw .469	cvm .440	ks .289	mad .192	α =.10
AD-19	4	ks .801	cvm .414	sw .413		.302 n = 10,
	20	ks 1.000	sw 1.000	cvm .999	mad .242	α =.10
AD-20	4	ks .609	cvm .219	sw .214	mad .094	.537 n = 20,
	20	ks 1.000	sw .954	cvm .944	mad .537	α =.10

CHAPTER 5

DISCUSSION

Recapitulation of Purpose and Design

This study has in principal part followed the model of a classical Monte Carlo study, in particular, for testing goodness of fit, informed by the tradition of such studies, employing simulations by computer, reported in the literature and treated in Chapter 2. The focus of this study has been a goodness-of-fit testing procedure devised by the researcher, with a test statistic denominated MADMASR and investigated in several respects. Using a range of sample sizes and pseudo-sampling with 100,000 replications, critical values were determined, with a focus on four representative sample sizes and four alpha values or Type I error rates. Other features of MADMASR were also recorded. The Type I error rates were verified with further sampling from exponential distributions. Then a representative sample of alternative probability distributions found in the relevant literature was employed to obtain rejection rates, again with 100,000 replications of each case. Lastly, performance of the researcher's goodness-of-fit statistic was compared with five other statistics representative of several decades of research on goodness-of-fit, and in particular, representative of alternative tests utilized in power studies in the literature on testing for exponentiality. The direct results (i.e., computer output) of the study, to wit, rejection rates and average statistics, are summarized in Tables 1 through 2, while Table VII characterizes the relative performance of the six goodness-of-fit tests, applied to the 20 alternative distributions (including one exponential distribution).

Indications of the Study

Overall, the researcher's goodness-of-fit test for exponentiality did not attain an acceptable level of power in respect to data from non-exponential populations, in the cases studied. There were a few exceptions under restrictive conditions, most notably, with alpha = .10, n = 20, for the uniform (AD-17) and Bernoulli mixture of Beta and Triangular distributions (AD-20), with power of .541 and .537 respectively. Generally, MADMASR did not achieve a level of power comparable to the highest powers achieved by the best performing tests among the various cases studied, that is, in the range of 80 or 90 percent or higher. It should be noted that such levels correspond to lowering the Type II error rate to 20 percent or less.

One of the variables in the present study was sample size, with simulations originally run for 42 values of n between 3 and 100 inclusive, but later narrowed to four values of n for the power study. The alternative distributions treated can be roughly separated according to the response to sample size, represented by the four sample sizes tested, n = 4, 10, 20, and 50 respectively. This response can be seen with most detail in Table 4. In some cases, power diminished consistently with increase among these four sample sizes. In other cases, power began to diminish after n = 10 or n = 20. Such a pattern may be related to the steady increase in critical values of MADMASR apparent in Table 1 and the increase in mean as well as standard deviation of 100,000 replications apparent in Table 2, as sample size advances from n = 3 to n = 100. While the procedure for computing the test statistic was designed to yield a value which was “standardized” in a generic sense, the distribution curve of the statistic rose with rising sample size. However, this may not be an impediment in itself for a goodness-of-fit sampling distribution. One of the most fundamental distributions for goodness-of-fit testing, Chi-Square, behaves in similar fashion, with both mean and variance increasing in nearly direct proportion to degrees of freedom.

For other alternative distributions than those with monotonically decreasing power, the power of the MADMASR test was stable or grew for n = 20 and fell dramatically for n = 50. The power results with respect to sample size may indicate value in refocusing the study on small to moderately small sample sizes, well below 50. Thus, a renewed study might attempt to locate more precisely, or establish a pattern, for the onset of marginal return for power, with increasing sample size. Upon suitable findings in that regard, further study might investigate a finer gradation of small to medium sample sizes. Given the leaps from 4 to 10 and from 20 to 50, especially, interesting patterns may have been missed in the current power study. See alternative distributions, AD-03, AD-06, AD-07, AD-08, AD-16, AD-17, and AD-20, and as well, AD-02 and AD-19, which peak when n = 10 among the four sample sizes tested. In regard to any of these, it may be asked how high power reaches, before it declines. Potentially, a sample size selection of n = 5, 10, 15, 20, 25, and 30, and possibly 35 and 40, would have been more revealing than the current study. Perhaps sizes below 10 should be omitted, since power above .1, approximately, was not in evidence for n = 4. In more focused research, perhaps increments in sample size as small as 1 or 2 would be worth study, as a preliminary investigation might help determine.

In a few of the alternative distributions, power of MADMASR climbed with increasing sample size (although this may not appear consistently for all alpha levels). See alternatives, AD-03 (the first gamma), AD-12 (Makeham), AD-15 (Chi-Square with 15 degrees of freedom), and AD-18 (half-normal). These distributions were among those few in which the researcher's statistic, while not notable under general power benchmarks, performed relatively well, among the goodness-of-fit statistics used for comparison.

There is some indication of increasing power, at least at moderate sample sizes, for Chi-Square populations with increasing degrees of freedom. However, additional study would require focus upon a more extensive selection from the Chi-Square family, to determine whether such a pattern can be corroborated and, if so, how significant an increase in power is achieved with greater degrees of freedom. Similarly, there is some suggestion that MADMASR may perform relatively well in regard to bell-shaped distributions, although the only evidence from this study is the brief progression of Chi-Square distributions, well below the level where normality is approximated, and the mixture of exponential and normal. Additional study could address bell-shaped distributions and the existence of any relationship between power of the researcher's test statistic and distributional parameters within the normal distribution family or parameters as well as higher moments distinguishing distributions which are asymptotically normal, such as Student's t and Chi-Square. For example, the skewness and kurtosis of a Chi-Square distribution, given by $\frac{2}{\sqrt{\lambda}}$ and $3\left(1+\frac{2}{\lambda}\right)$ respectively for λ degrees of freedom approach zero and 3 respectively, the values for any normal variate, as λ increases (Bury, 1999, pp. 210, 213). The results of such a study might reflect on the quality of standardization attempted in the MADMASR procedure.

Another area of potential for the MADMASR statistic is mixture distributions. Only four of these of distinct types were utilized in this study. A more focused investigation would address a range of similarly constructed mixture distributions, varying the parameters of component populations and the proportions in which they are mixed. Those denominated "anomalously shaped" distributions here (AD-19 and AD-20) were predefined finite populations, pseudo-randomly constructed from functionally-defined densities, unlike the first set of mixture distributions (AD-01 and AD-02) with variates constructed anew from functionally-defined densities for each replication. The underlying notion was that data in the material or sensible, observable world is less "well-formed" than the common statistical density functions and more nebulous in respect to revealing any visible structure. Future studies could methodically examine facsimiles of such data sets with a structured progression of characteristics (if the inherent paradox in such a simulation can be transcended.)

The researcher’s goodness-of-fit procedure appeared to show some promise in its sensitivity respecting the Makeham and Weibull distributions, useful in survival and reliability analysis. Other distributions related to survival or reliability are those with a “bathtub” shaped failure rate distribution (see Barlow & Proschan, 1981, p. 55). If support is restricted to the unit interval, the Beta distribution family can attain a parametrically-controlled variety of bathtub-like and other shapes, a fact which motivated utilizing Beta distributions in the mixture distributions, AD-19 and AD-20. One possible study might explore goodness-of-fit, applied to Beta alternatives selected methodically so as to focus patterns which would not have been revealed in the current study.

Limitations of the Study

An implicit, concomitant theme of this study was the investigation of a broad variety of distributions. First, it was necessary to present a test for goodness-of-fit with alternative distributions. As discussed previously in this Chapter, no single parametric or operationally-defined class of distributions was identified nor, therefore, was submitted to a methodical study. Rather, a selective set of alternative distributions was employed, based upon those frequently appearing in the literature on goodness-of-fit testing. In addition to the limitations due to this selectivity, the description of the distributions, as treated here, was limited to the first two moments.

Other factors, particularly shape, would be relevant to goodness-of-fit. This limitation relates to the researcher's statistic, insofar as the characteristic shape expected of an exponential distribution did not partake in the calculation of that statistic. This could be remedied by using a statistic weighted according to considerations of the relative density of points expected in different regions of the domain. A related issue involves the use of linear regression in obtaining the MADMASR statistic, whereas the null distribution is non-linear, specifically, exponential. It is possible that exploring the use of non-linear regression for calculating residuals and, ultimately, MADMASR, could lead to improvement of the peformance of this goodness-of-fit statistic.

Another question involving distributional attributes is how a goodness-of-fit statistic would be affected by inclusion of an additional parameter. With the exponential distribution, a second, location parameter may be estimated by the minimum observation in a sample being tested for two-parameter exponentiality. This may not have a beneficial result for omnibus testing, since permitting another dimension of variation for the null distribution may merely allow for a wider range of non-exponential alternatives to present features resembling exponentiality. This issue, however, was not treated in the current study and might be worth investigating in any study entailing a more focused set of alternatives, such as members of a single parametric family or methodically varied results of a single distribution-mixing algorithm, or members of a restricted class of failure rate distributions.

There was evidence of numerical difficulties, resulting in underflow in the researcher's Monte Carlo subroutines. The effect of computational anomalies in the course of thousands of iterations of arithmetically intensive procedures is not easily understood and may, indeed, serve as the topic itself of Monte Carlo experimentation. Such experimentation might refine the procedures used here, but was not included in this study. Moreover, the failure to completely resolve such matters may have afflicted the study in some unpredictable manner.

Investigation of the MADMASR statistic was restricted to right-tailed testing, insofar as maximum deviations were involved. Should any illumination have been available from examining the left-tail as well, that was excluded here. A related issue concerns the conventions of goodness-of-fit testing typified in a Chi-Square test. That is, the alternative hypothesis is deemed to be supported by evidence of a small p-value. However, consistency with the null hypothesis may be quantified as well, with a sufficiently large p-value, although this lies outside the traditional dichotomy of rejecting or failing to reject the null hypothesis. At least in a descriptive sense, inclusion of upper-tail areas of 20 percent or more may reveal interesting features in the performance of goodness-of-fit statistics under study.

Programming itself presented limitations in this study. Some goodness-of-fit tests, with the increasing sophistication of statistical theory over several decades, are computationally and algorithmically complex. In a portion of the literature, moreover, the primary issue is asymptotic behavior of a goodness-of-fit statistic. The alternative tests presented here were, by necessity, those found to be amenable to translation into a computable task which itself would be repeated millions of times in Monte Carlo processing. Another factor in some Monte Carlo programming not enlisted here is a variable stopping procedure, in lieu of a fixed number of replications. Such a procedure would entail internal testing to achieve a desirable degree of variance reduction. The value of such a procedure for the current study was not investigated.

Future Research And Other Realms Of Research

Various areas of statistical research, some of which present attractive topics for future study, were examined. The theoretical background of what is currently known as information theory was available in the second half of the Twentieth Century but applications waited, not only for the tremendously enhanced computational power of recent decades but also for cross-fertilization between distinct realms of applied mathematical and statistical theory, Information theory was the source of the KLC statistic utilized here in the comparative power study. This remains a vital area for contemporary research.

Another area is that of mixture distribution models. While their involvement here was constructive with the model determined, in much research with mixture models, the problem is one of fitting a model to data (see, for example, Bőhning, 2000). Model fitting is the inverse problem to the attempt in this study to construct known mixture distributions to mimic ill-defined, real world data. A taxonomy of even simply constructed mixture distributions may include distributions the statistical attributes of which are not treated in the literature and thus present a challenge for future research utilizing Monte Carlo methods.

One source of less tractable tests for Monte Carlo study is differential geometry and, more generally, the concept of distance. The researcher did explore this area prior to arriving at the testing procedure in this study. The area is as attractive as it is elusive in articulating a bridge from distribution theory to analysis of finite data samples.

Two goodness-of-fit procedures in the more recent literature involve the Lorentz curve and Gini index, originated in the economic theory of income and wealth distributions. The Gini statistic, endorsed as the best of the two in most of the literature, was utilized here, with limited success relative to the other tests employed. In relation to more focused testing for particular distributions, this statistic may have unrealized potential worth exploring.

The diversity of probability density functions is a topic closely, if tangentially associated with this investigation. Reliability theory and survival, the analogous study within biological realms, as well as size distributions of income and wealth are pragmatic realms, among others, in which probability densities have been developed and studied, proliferating beyond the relatively small class of distributions encountered in the mainstream of statistical methods used in data analysis. Among these are distributions which may not be fully described in the literature and thus present a challenge for future Monte Carlo research.

APPENDIX A: LAMBDA STAR AND THE THETA STAR ESTIMATE

We begin by observing that a finite random sample from the exponential distribution will have sample mean and standard deviation which are not exactly the same, both due to sampling error and due to the fact that a finite, discrete set of iid variates cannot mimic precisely the population distribution of a continuous random variable. Indeed, to assert a third rationale, although it is dominated by sampling variability, the sample standard deviation is a biased statistic, although the sample mean is not. Then, excepting the unlikely cases of sample statistics which, in approximation, are equal in value, there are two possibilities: 1) $\overline{x} < s$ and 2) $s < \overline{x}$. We shall position an estimate of the exponential parameter, θ, which equals both population mean and standard deviation, between the two statistics. This presents two cases, expressed by the following schema:

case 1) $\overline{x}.....\hat{\theta}...s$ and case 2) $s...\hat{\theta}.....\overline{x}$.

Corresponding to these cases, we assign a number, λ, such that $0 < \lambda < 1$, yielding the formulas: case 1) $\theta = \overline{x} + \lambda\left(s - \overline{x}\right)$ and case 2) $\theta = s + \lambda\left(\overline{x} - s\right)$.

(If we were assuming a two-parameter exponential distribution, the difference between the sample mean and sample standard deviation could be taken as an estimate of a second, location parameter, μ, in a density function of the form $f\left(x;\, \mu,\theta\right) = \frac{1}{\theta}\exp\left[-\frac{x-\mu}{\theta}\right],\ x \geq \mu \geq 0$, but the null distribution for the purposes of this study is the simple exponential, with $f\left(x;\, \theta\right) = \frac{1}{\theta}\exp\left[-\frac{x}{\theta}\right]$. See Bury, 1999, p. 178.)

We shall seek a particular estimate of θ, called θ^* (theta-star), which minimizes the sum of squared deviations between the empirical distribution and the projected simple exponential distribution with parameter equal to θ^*. The value of λ which yields $\theta = \theta^*$ is called λ^* (lambda-star). To maintain the formulas in tractable form, we shall determine lambda-star by comparing the sample data, $x_1,...,x_n$ with the inverse cumulative exponential distribution applied to the empirical cumulative values, in the form $\frac{2i-1}{2n},\ i = 1,...n$.

The inverse cdf of the simple exponential distribution is easily found, letting $y \;=\; F\left(x;\, \theta\right) \;=\; \int_0^x \frac{1}{\theta}\exp\left[-\frac{t}{\theta}\right]dt \;=\; 1-\exp\left[-\frac{x}{\theta}\right]$ and solving for x, we obtain $\ln\left(1-y\right) \;=\; -\frac{x}{\theta}$, and $F^{-1}\left(y;\, \theta\right) \;=\; x \;=\; -\theta\ln\left(1-y\right)$. Thus, we seek to minimize $\sum_{i=1}^{n}\left\{x_i - F^{-1}\left(\frac{2i-1}{n};\, \theta^*\right)\right\}^2 \;=\; \sum_{i=1}^{n}\left\{x_i - \left[-\theta^*\ln\left(1-\frac{2i-1}{n}\right)\right]\right\}^2$. Although we need to evaluate lambda-star, upon which theta-star depends, the procedure will be developed in terms of theta-star, in order to account for both cases described above, noting that in case 1), $\frac{d\theta^*}{d\lambda^*} = s - \bar{x}$ and in case 2), $\frac{d\theta^*}{d\lambda^*} = \bar{x} - s$.

$$\text{Let } \phi = \sum_{i=1}^{n}\left\{ x_i + \theta^* \ln\left(1 - \frac{2i-1}{n}\right)\right\}^2 = \sum_{i=1}^{n}\left\{ x_i + \theta^* \ln\left(w_i\right)\right\}^2$$

$= \sum_{i=1}^{n} x_i^2 + 2\theta^* \sum_{i=1}^{n} x_i \ln\left(w_i\right) + \theta^{*2} \sum_{i=1}^{n} \left[\ln\left(w_i\right)\right]^2$ and set $\frac{d\phi}{d\lambda^*} = \frac{d\phi}{d\theta^*}\frac{d\theta^*}{d\lambda^*}$ equal to zero. Then $\frac{d\phi}{d\lambda^*} = 2\left\{\sum_{i=1}^{n} x_i \ln\left(w_i\right) + \theta^* \sum_{i=1}^{n}\left[\ln\left(w_i\right)\right]^2\right\}\left[\left(s - \bar{x}\right)\cdot sign\left(s - \bar{x}\right)\right] = 0$. If the sample statistics are, atypically, equal, then they trivially yield theta-star.

Otherwise, $\theta^* = -\frac{\sum_{i=1}^{n} x_i \ln\left(w_i\right)}{\sum_{i=1}^{n}\left[\ln\left(w_i\right)\right]^2}$, which suffices for both cases 1) and 2).

APPENDIX B: FORTRAN CODE FOR MONTE CARLO PROCEDURES

The FORTRAN programs and module listed in this Appendix are those which were created by the researcher specifically for this Monte Carlo study. They include the following source code listings in Lahey Essential Fortran 90, version 4.0, created and maintained as files with "f90" extension, within the Lahey ED editing environment, on the researcher's office and home computers. The module, RANGEN, furnished by the College of Education to students for Monte Carlo-based research, is not listed. Each program or module begins on a new page.

1. program MCexptst01_outproc (outputs Table 1)
2. program MCexpontest01 (outputs Table 2)
3. program mcexp_verify (outputs Table 3)
4. program mcexp_rejectrates (outputs Table 4)
5. program mcexptst_pwr (outputs Table 5)
6. module utils4tstexpdst (includes subroutines utilized by the programs above)

```
!    Last change:  AJT  16 Feb 2006    3:16 pm
program MCexptst01_outproc
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!  Programming of MCexptst01_outproc
!  was begun on March 28, 2004
!            by
!      Andrew Tierman
!
!  Program to process output from MCexpontest01 ----
!  that output consisting of Monte Carlo samples of a test statistic
!  for Exponentiality of an empirical distribution  under the null
!  that sampling was from an (unspecified, one-parameter)
!  exponential distribution. OUTPUT: TABLE 1
!
!  THIS PROGRAM CONSTRUCTS A CRITICAL VALUE TABLE FOR
!  TAIL AREAS, .01, .025, .05 and .1, FOR ALL SAMPLE SIZES
!  STACK SIZE: 405400 Bytes NEEDED - RUN THE FOLLOWING COMMAND
!  IN DOS BEFORE RUNNING EXECUTABLE:
!  elf90 mcexptst01_outproc  -stack 410000
!
!      Revised Date: April 4-10, 2005
!      Additions: July 6-7, 2005  -  critical values of MADMASR
!         for selected sample sizes
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!###############################################################
!  EXTERNAL PROGRAMS REFERENCED >>>>>
!###############################################################
USE rangen
USE utils4tstexpdst
implicit none
!###############################################################
!          TYPE DECLARATIONS
!###############################################################
REAL(KIND=8)::time1,time2,runtime
REAL,ALLOCATABLE,DIMENSION(:)::xdata
REAL::zero,one
REAL(KIND=8)::dseed
REAL::ytestat(100001)
REAL::lamstar,thestar,medabsresid,madresid
REAL::crit_4prcntls(4),rgt_crit_4prcntls(4),empir_4prcntls(4)
REAL::table1(42,4),table1a(4,4)                    !7-6-05
INTEGER::samplsiz(42),samplsiz_4select(4)                 !7-6-05
INTEGER::i,j,n,tenthou
CHARACTER(LEN=3)::fn_n(42)
CHARACTER(LEN=14)::filename
CHARACTER(LEN=50)::line
!###############################################################
!      PROGRAM RUN TIMING;
!      VALUE INITIALIZATION;
!      OPEN TABULAR OUTPUT TEXT FILE;
!      ARRAY ALLOCATION
!###############################################################
call cpu_time(time1)
```

```
zero=0.0
one=1.0
! change from 10,000 to 100,000 repetitions
tenthou=100000
!tenthou=1000 ! TEST OF LOWER REPS  JULY 18, 2005 ***********************************
!dseed = tenthou*tenthou*COS(time1)
dseed = 1.1111D9*COS(time1)
call cpu_time(time2)
dseed = dseed + 7777.0*SIN(time2)
crit_4prcntls(:)=(/.01,.025,.05,.1/)
samplsiz_4select(:)=(/2,8,18,32/)                    !7-6-05
do i=1,4
j=4-i+1
rgt_crit_4prcntls(i)=one-crit_4prcntls(j)
end do
!samplsiz(:)=(/4,8,16,32,64/) ORIGINAL PILOT HAD 5 SAMPLE SIZES
!   SET UP 42 SAMPLE SIZES: 3-30 by units and 35-100 by fives
do i=1,28
samplsiz(i)=i+2
end do
do i=29,42
samplsiz(i)=30+5*(i-28)
end do
!
!          OUTPUT FILES HAVE NAMES ENDING IN THE SAMPLE SIZE OF THE
!          SAMPLES REPRESENTED. ANOTHER PROGRAM WILL SORT AND
!          ARRAY THIS OUTPUT
!
!fn_n(:)=(/'04','08','16','32','64'/)   ORIGINAL PILOT HAD 5 SAMPLE SIZES
!PREFIX 'A' BEFORE SAMPLE SIZE IN FILE NAME IS FOR MAD STATISTIC MONTE CARLO
DATA
fn_n(:)=(/'003','004','005','006','007','008','009','010',&
&'011','012','013','014','015','016','017','018','019','020',&
&'021','022','023','024','025','026','027','028','029','030',&
&'035','040','045','050','055','060','065','070','075','080','085','090','095','100'/)
!
!      OPEN OUTPUT FILE FOR TABULAR TEXT, TABLE # 1
!
OPEN(199,FILE='exptst_table1.txt',status='old')
!##########################################################
!
!        OPEN SAMPLE ARRAY AND SAMPLE FILE
!        INSIDE LOOP FOR SAMPLE SIZES
!##########################################################
do i=1,42
n=samplsiz(i)
ALLOCATE (xdata(n))
filename='exptst'//'A'//fn_n(i)//'.out'
OPEN(i,FILE=filename,STATUS='old')
WRITE(*,*)'opening output file: ',filename,' sample no. ',i,' has n = ',n   !DIAGNOSTIC
!##########################################################
!              ""   LOOP FOR SAMPLES
!##########################################################
```

```
!   READ OUTPUT FILE
do j=1,tenthou
READ(i,FMT="(f20.9)")ytestat(j)
!do j=1,tenthou
!          ******** get random sample from exponential
!                   distribution, using rangen ***********
!
!       call exp1(dseed,n,one,xdata)
!      !WRITE(*,*)xdata(:)      !!!!!!!!!!DIAGNOSTIC PRINT
!       call expon_star(n,xdata,lamstar,thestar,medabsresid,madresid)
!      ytestat(j)=madresid
!       !WRITE(*,*)ytestat(j)   !!!!!!!!!!DIAGNOSTIC PRINT
!       WRITE(i,FMT="(f20.9)")ytestat(j)
end do
call fastsort(tenthou,ytestat)
!WRITE(*,*)ytestat(:)    !DIAGNOSTIC
call rgt_tld_crt_val4(tenthou,ytestat,crit_4prcntls,empir_4prcntls)
table1(i,:)=empir_4prcntls(:)
DEALLOCATE(xdata)
END do
!###########################################################################
!        Set Array of Critical Values for 4 Selected n's (4, 10, 20, 50)
!###########################################################################
table1a(:,:) = table1(samplsiz_4select,:)                    ! 7-7-05
OPEN(200,FILE='exptst_4by4cvarray1a.txt',status='old')         ! 7-7-05
do i = 1,4                                             ! 7-7-05
WRITE(200,FMT="(4F15.12)")table1a(i,:)                        ! 7-7-05
WRITE(*,*)table1a(i,:)                                 ! 7-7-05
END do                                                 ! 7-7-05
CLOSE(200)                                             ! 7-7-05
!WRITE(*,*)' after close 200'                          ! 7-7-05
!OPEN(200,FILE='exptst_4by4cvarray1a.txt',status='old')         ! 7-7-05
!WRITE(*,*)'       after open 200 after close 200'              ! 7-7-05
!do i = 1,4                                            ! 7-7-05
!read(200,FMT="(4F15.12)")table1a(i,:)                         ! 7-7-05
!WRITE(*,*)table1a(i,:)                                ! 7-7-05
!END do                                                ! 7-7-05
!##############################################################
!
!        P R I N T   T A B L E  1
!
!##############################################################
WRITE(*,*)
line='__________________________________________________'
WRITE(199,FMT="(36X,'TABLE 1')")
WRITE(*,FMT="(36X,'TABLE 1')")
WRITE(199,*)
WRITE(*,*)
WRITE(199,FMT="(14X,A50)")line
WRITE(*,FMT="(14X,A50)")line
WRITE(199,*)
WRITE(*,*)
!         Title of table: Critical Percentiles of the Maximum
```

```
!           Absolute Deviation from the Median of the theta-star-
!           Standardized Residuals ( ( x_i - med )/theta-star )
!           From Regression of theta-star Inverse cdf on
!           the Empirical Distribution
!
WRITE(199,FMT="(19X,'Critical Percentiles of the Maximum Absolute')")
WRITE(199,FMT="(19X,'Deviation from the Median of the theta-')")
WRITE(199,FMT="(19X,'star-standardized Residuals from Regres-')")
WRITE(199,FMT="(19X,'sion of the theta-star Inverse CDF on the')")
WRITE(199,FMT="(19X,'Empirical Distribution, for sample sizes')")
WRITE(199,FMT="(19X,'between 3 and 100; 100,000 replications')")
WRITE(199,*)
!         SCREEN PRINT
WRITE(*,FMT="(19X,'Critical Percentiles of the Maximum Absolute')")
WRITE(*,FMT="(19X,'Deviation from the Median of the theta-')")
WRITE(*,FMT="(19X,'star-standardized Residuals from Regres-')")
WRITE(*,FMT="(19X,'sion of the theta-star Inverse CDF on the')")
WRITE(*,FMT="(19X,'Empirical Distribution, for sample sizes')")
WRITE(*,FMT="(19X,'between 3 and 100; 100,000 replications')")
WRITE(*,*)
!
!WRITE(199,FMT="(21X,'Critical Values of MAD, for alpha =')")  ORIG. TABLE TITLE
!WRITE(*,FMT="(21X,'Critical Values of MAD, for alpha =')")    ORIG. TABLE TITLE
WRITE(199,FMT="(15X,'n',4F12.3)")rgt_crit_4prcntls(:)
WRITE(*,FMT="(15X,'n',4F12.3)")rgt_crit_4prcntls(:)
WRITE(199,FMT="(14X,A50)")line
WRITE(*,FMT="(14X,A50)")line
!        BODY OF TABLE 1
do i=1,42
WRITE(199,FMT="(10X,I6,4F12.4)")samplsiz(i),table1(i,:)
WRITE(*,FMT="(10X,I6,4F12.4)")samplsiz(i),table1(i,:)
end do
WRITE(199,FMT="(14X,A50)")line
WRITE(*,FMT="(14X,A50)")line
WRITE(*,*)
!
!
!   DIAGNOSTIC PRINTS TO SCREEN:(provide information on statistics used globally)
!WRITE(1,FILE='chkmth',STATUS='new')y
!call expon_star(5,y,lamstar,thestar,medabsresid,madresid)
!WRITE(*,*)' lambda star = ',lamstar
!WRITE(*,*)' theta star = ', thestar
!WRITE(*,*)' median absolute residual = ', medabsresid
!RITE(*,*)' median absolute deviation = ',madresid
!WRITE(*,*)
call cpu_time(time2)
runtime=time2-time1
WRITE(*,*)' the runtime was ',runtime
stop
end program MCexptst01_outproc
```

```
!    Last change:  AJT  16 Feb 2006    3:20 pm
program MCexpontest01
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
!  Programming of MCexpontest01    >>>>
!  was begun on March 16, 2004     >>>>
!           by                  >>>>
!      Andrew Tierman             >>>>
!  This Program will simulate      >>>>
!  the sampling distribution       >>>>
!  of a test statistic for        >>>>
!  Exponentiality of an           >>>>
!  empirical distribution,        >>>>
!  under the null, i.e.           >>>>
!  sampling from the              >>>>
!  exponential dist.              >>>>
!  Revised April 4-10, 2005       >>>>
!  OUTPUT: Produces Table 2       >>>>
!                             >>>>
!  STACK SIZE: 806216 Bytes NEEDED >>>>
!  RUN THE FOLLOWING COMMAND IN DOS>>>>
!  BEFORE RUNNING EXECUTABLE:      >>>>
!  elf90 mcexpontest01  -stack 810000
!                             >>>>
!<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<
!  EXTERNAL PROGRAMS REFERENCED >>>>>>>>
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
USE rangen
USE utils4tstexpdst
implicit none
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
!  TYPE DECLARATIONS        >>>>>>
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
REAL(KIND=8)::time1,time2,runtime
REAL,ALLOCATABLE,DIMENSION(:)::xdata
REAL::zero,one
REAL(KIND=8)::dseed
REAL::ytestat(100001),thetastars(100001),table2(42,9),statarray(8)
REAL::xbar,sd,skew,kurt
REAL::lamstar,thestar,medabsresid,madresid,maxabsdevfrmed
INTEGER::samplsiz(42)
INTEGER::i,j,n,tenthou
CHARACTER(LEN=3)::fn_n(42)
CHARACTER(LEN=14)::filename
CHARACTER(LEN=65)::line
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
!  TIMING; VALUE INITIALIZATION; FILES; ARRAY ALLOCATION  <<
!>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
call cpu_time(time1)
zero=0.0
one=1.0
! change repetitions from 10,000 to 100,000
tenthou=100000
!tenthou=1000 ! TEST OF LOWER REPS  JULY 18, 2005 **********************************
```

```
!dseed =(239847+time1)*tenthou*COS(time1)
dseed = 1.1111D9*COS(time1)
call cpu_time(time2)
dseed = dseed + 7777.0*SIN(time2)
!#########################################################
!###########   OPEN TABLE 2 OUTPUT FILE        ###########
!#########################################################
OPEN(299,FILE='exptst_table2.txt',status='old')
!samplsiz(:)=(/4,8,16,32,64/)  ORIGINAL PILOT HAD 5 SAMPLE SIZES
!CURRENT: files 1-28 have n = 3,4..30; files 29-42 have n = 35,40..100
!
do i=1,28
samplsiz(i)=i+2
end do
do i=29,42
samplsiz(i)=30+5*(i-28)
end do
!   OUTPUT FILES HAVE NAMES ENDING IN THE SAMPLE SIZE OF THE
!   SAMPLES REPRESENTED. ANOTHER PROGRAM WILL SORT AND
!   ARRAY THIS OUTPUT
!fn_n(:)=(/'04','08','16','32','64'/)   ORIGINAL PILOT HAD 5 SAMPLE SIZES
fn_n(:)=(/'003','004','005','006','007','008','009','010',&
&'011','012','013','014','015','016','017','018','019','020',&
&'021','022','023','024','025','026','027','028','029','030',&
&'035','040','045','050','055','060','065','070','075','080','085','090','095','100'/)
!##########################################################
!               LOOP FOR SAMPLE SIZES
!##########################################################
do i=1,42          !42 REGULARLY, 15 ONLY FOR DIAGNOSTIC PURPOSES (SKEW AND
KURT)
n=samplsiz(i)
ALLOCATE (xdata(n))
filename='exptst'//'A'//fn_n(i)//'.out'
OPEN(i,FILE=filename,STATUS='old')
WRITE(*,*)'opening output file: ',filename,i,j,n   !DIAGNOSTIC
!##########################################################
!              """""""""""""""  LOOP FOR SAMPLES
!##########################################################
do j=1,tenthou
!          ******** get random sample from exponential
!                  distribution, using rangen ***********
!
        call exp1(dseed,n,one,xdata)
       !WRITE(*,*)xdata(:)      !!!!!!!!!!DIAGNOSTIC PRINT
        call expon_star(n,xdata,lamstar,thestar,medabsresid,madresid,maxabsdevfrmed)
        ytestat(j)=maxabsdevfrmed                         !!!!! ORIG: madresid
        thetastars(j)=thestar
        !WRITE(*,*)ytestat(j)   !!!!!!!!!!DIAGNOSTIC PRINT
        WRITE(i,FMT="(f20.9)")ytestat(j)
end do
!#########################################################################
############################
!   Compute Moments of ytestat and thetastars after replications, for each
```

```
!   Sample Size and print in table2: sample mean, std dev, skewness, kurtosis
!###############################################################################
############################
!   table2(42,9) stores for each sample size the four statistics from four_stats, for theta star and
ystat
!###############################################################################
############################
        call four_stats(tenthou,thetastars,xbar,sd,skew,kurt)
        statarray(1:4)=(/xbar,sd,skew,kurt/)
        call four_stats(tenthou,ytestat,xbar,sd,skew,kurt)
        statarray(5:8)=(/xbar,sd,skew,kurt/)
        table2(i,1)=i
        table2(i,2:9)=statarray(1:8)
        !DIAGNOSTIC PRINT   SKEWNESS, KURTOSIS
        !WRITE(*,*)SKEW,KURT
DEALLOCATE(xdata)
END do
!##########################################################
!  PRINT TABLE TWO to file
!##########################################################
write(*,*)
line='_________________________________________________________________'
WRITE(299,FMT="(36x,'TABLE 2')")
WRITE(*,FMT="(36x,'TABLE 2')")
WRITE(299,*)
WRITE(*,*)
WRITE(299,FMT="(10X,A65)")line
WRITE(*,FMT="(10X,A65)")line
WRITE(299,*)
WRITE(*,*)
WRITE(299,FMT="(18X,'Statistics, by sample size, for 100,000 replications of')")
WRITE(299,FMT="(15X,'the estimator, theta-star, and the Maximum Absolute Deviation')")
WRITE(299,*)
WRITE(*,FMT="(18X,'Statistics, by sample size, for 100,000 replications of')")
WRITE(*,FMT="('the estimator, theta-star, and the Maximum Absolute Deviation')")
WRITE(*,*)
WRITE(299,FMT="(13X,' - parameter estimate -',13X,' - Max. Abs. Dev. statistic -')")
WRITE(*,FMT="(13X,' - parameter estimate -',13X,' - Max. Abs. Dev. statistic -')")
WRITE(299,*)
WRITE(*,*)
WRITE(299,FMT="(3X,'n',4X,2('mean',3x,'std.dev.',1x,'skewness',1x,'kurtosis',6x))")
WRITE(*,FMT="(3X,'n',4X,2('mean',3x,'std.dev.',2x,'skewness',2x,'kurtosis',4x))")
WRITE(299,FMT="(10X,A65)")line
WRITE(*,FMT="(10X,A65)")line
do i=1,42
WRITE(299,FMT="(1X,I3,4F9.3,3X,4F9.3)")samplsiz(i),table2(i,2:)
WRITE(*,FMT="(1X,I3,4F9.3,3X,4F9.3)")samplsiz(i),table2(i,2:)
end do
WRITE(299,*)
WRITE(*,*)
WRITE(299,FMT="(10X,A65)")line
WRITE(*,FMT="(10X,A65)")line
!
```

```
call  cpu_time(time2)
runtime=time2-time1
WRITE(*,*)' the runtime was ',runtime
stop
end program MCexpontest01
```

```
!    Last change:  AJT  20 Jul 2005    4:59 pm
program mcexp_verify
!++++++++++++++++++
! Programming begun July 7, 2005
! by  Andrew Tierman
!
! This program reads the four upper critical values
! stored in mcexptst01_outproc for four selected sample
! sizes. The critical values are used to find
! TYPE I error rate (percent of rejections for pseudo-
! random samples from a one-parameter exponential dist-
! tribution with theta = 0.1, 0.5, 2.0 and 10.0, res-
! pectively.
!
! This program outputs Table Three
!++++++++++++++++++
!
!  EXTERNAL PROGRAMS REFERENCED
!
!++++++++++++++++++
USE rangen
USE utils4tstexpdst
implicit none
!++++++++++++++++++
!
!  TYPE DECLARATIONS
!
!++++++++++++++++++
REAL(KIND=8)::time1,time2,runtime
REAL,ALLOCATABLE,DIMENSION(:)::xdata
REAL::zero,one,tru_theta
REAL(KIND=8)::dseed
REAL::ytestat(100001),thetastars(100001)
REAL::alpha(4),exp_param(4),rejectrate(4)
REAL::lamstar,thestar,medabsresid,madresid,maxabsdevfrmed
REAL::xbar,sd,skew,kurt
REAL::rgt_crit_4prcntls(4),empir_4prcntls(4)
REAL::statarray(8)
REAL::table3(4,4,6) ! OP array: indices for parameter, sample size & alpha respectively
REAL::table1a(4,4)  ! CV array: indices for selected sample size & alpha (from Table1)
INTEGER::selsamsiz(4)
INTEGER::i,j,k,n,tenthou
CHARACTER(LEN=3)::fn_n(42)      !????
CHARACTER(LEN=14)::filename     !????
CHARACTER(LEN=76)::line
!+++++++++++++++++
!
!  PROGRAM RUN TIMING;
!  VALUE INITIALIZATION;
!  OPEN TABULAR OUTPUT TEXT FILE;
!  ARRAY ALLOCATION
!
!++++++++++++++++++
```

```
call cpu_time(time1)
zero=0.0
one=1.0
tenthou = 100000  ! chgd to 10^5 repetitions, preserving "tenthou" for all program modules
!tenthou=1000 ! TEST OF LOWER REPS  JULY 18, 2005 **********************************
dseed = 1.1111D9*COS(time1)
call cpu_time(time2)
dseed = dseed + 7777.0*SIN(time2)
line='__________________________________________________________________________________
___'
exp_param(:) = (/0.1, 0.5, 2.0, 10.0/)
selsamsiz(:) = (/4, 10, 20, 50/)
rgt_crit_4prcntls(:)=(/.90, .95, .975, .99/)
alpha(:) = one - rgt_crit_4prcntls(:)
OPEN(399,FILE='exptst_table3.txt',status='old')
!+++++++++++++++++++
!
!  OPEN FILE AND READ CRITICAL
!  VALUES FOR SELECTED SAMPLE SIZES,
!  n = 4, 10, 20, 50, FROM TABLE 1
!
!+++++++++++++++++++
OPEN(200,FILE='exptst_4by4cvarray1a.txt',status='old')  ! subscripts are for n and rt-tld-%ile
do i = 1,4
READ(200,FMT="(4F15.12)")table1a(i,:)
END do
!+++++++++++++++++++
!
! P R I N C I P A L   L O O P S  for  M O N T E   C A R L O
!
!+++++++++++++++++++
! L O O P   for  E X P O N E N T I A L   P A R A M E T E R S
!+++++++++++++++++++
do i = 1,4          ! ECTO
!+++++++++++++++++++
tru_theta = exp_param(i)
!+++++++++++++++++++
! L O O P   for  S A M P L E   S I Z E S
!+++++++++++++++++++
do j = 1,4          ! MESO
!+++++++++++++++++++
n = selsamsiz(j)
ALLOCATE (xdata(n))
!+++++++++++++++++++
! L O O P  for  R E P L I C A T I O N S
!+++++++++++++++++++
do k = 1,tenthou     ! ENDO
!+++++++++++++++++++
call exp1(dseed,n,tru_theta,xdata)
call expon_star(n,xdata,lamstar,thestar,medabsresid,madresid,maxabsdevfrmed)
ytestat(k) = maxabsdevfrmed
thetastars(k) = thestar
!+++++++++++++++++++
```

```
end do              ! ENDO  next k - replication
!+++++++++++++++++++
call four_stats(tenthou,thetastars,xbar,sd,skew,kurt)
statarray(1:4) = (/xbar, sd, skew, kurt/)
WRITE(*,*)' tru_theta = ',tru_theta,'   mean thetastar = ',xbar
call four_stats(tenthou,ytestat,xbar,sd,skew,kurt)
statarray(5:8) = (/xbar, sd, skew, kurt/)
call fastsort(tenthou,ytestat)
call upr_rej_rates(ytestat, tenthou, table1a(j,:), rejectrate)
table3(i,j,1:4) = rejectrate(:)
table3(i,j,5:6) = statarray(1:2)
DEALLOCATE(xdata)
!+++++++++++++++++++
end do              ! MESO  next j - sample size
!+++++++++++++++++++
!+++++++++++++++++++
end do              ! ECTO  next i - true sampled theta
!+++++++++++++++++++
!+++++++++++++++++++
!
!       P R I N T   T A B L E  3
!
!+++++++++++++++++++
WRITE(*,*)
WRITE(399,FMT="(36X,'TABLE 3')")
WRITE(*,FMT="(36X,'TABLE 3')")
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(3X,A76)")line
WRITE(*,FMT="(3X,A76)")line
WRITE(399,*)
WRITE(*,*)
!         working Title of table three:   Type I error rates under sampling
!             distribution of the test statistic, MADMASR,
!              computed for 100,000 replications each, with random sampling for four sample
!                 sizes each for four exponential distributions, theta = ...
!
WRITE(399,FMT="(19X,'TYPE I ERROR RATES, on 100,000 replications')")
WRITE(399,FMT="(15X,'with simulated random sampling from four exponential')")
WRITE(399,FMT="(15X,'distributions with parameter equal to 0.1, 0.5, 2 and 10')")
WRITE(399,FMT="(15X,'respectively, each with sample sizes of 4, 10, 20, and 50,')")
WRITE(399,FMT="(15X,'for right-tailed alpha values of .10, .05, .025, and .01')")
WRITE(399,FMT="(15X,'The mean and std. dev. of the estimated exponential parameter')")
WRITE(399,FMT="(15X,'for all replications, are also shown for each case.')")
WRITE(399,*)
!          SCREEN PRINT
WRITE(*,FMT="(19X,'TYPE I ERROR RATES, on 100,000 replications')")
WRITE(*,FMT="(15X,'with simulated random sampling from four exponential')")
WRITE(*,FMT="(15X,'distributions with parameter equal to 0.1, 0.5, 2 and 10')")
WRITE(*,FMT="(15X,'respectively, each with sample sizes of 4, 10, 20, and 50,')")
WRITE(*,FMT="(15X,'for right-tailed alpha values of .10, .05, .025, and .01')")
WRITE(*,FMT="(15X,'The mean and std. dev. of the estimated exponential parameter')")
WRITE(*,FMT="(15X,'for all replications, are also shown for each case.')")
```

```
WRITE(*,*)
!
WRITE(399,FMT="(3X,A76)")line
WRITE(*,FMT="(3X,A76)")line
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(3X,'alpha',5X,4F11.3,7X,'theta-star')")alpha(:)
WRITE(*,FMT="(3X,'alpha',5X,4F11.3,7X,'theta-star')")alpha(:)
WRITE(399,FMT="(3X,'theta',4X,'n',50X,'(est''d param.)')")
WRITE(*,FMT="(3X,'theta',4X,'n',50X,'(est''d param.)')")
WRITE(399,FMT="(64X,'mean',4X,'std.dev.')")
WRITE(*,FMT="(64X,'mean',4X,'std.dev.')")
WRITE(399,FMT="(3X,A76)")line
WRITE(*,FMT="(3X,A76)")line
!        BODY OF TABLE 1
do i=1,4
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(1X,f6.1)")exp_param(i)
WRITE(*,FMT="(1X,f6.1)")exp_param(i)
do j=1,4
WRITE(399,FMT="(9X,I4,1X,4F11.4,2X,2F9.3)")selsamsiz(j),table3(i,j,1:4),table3(i,j,5:6)
WRITE(*,FMT="(9X,I6,1X,4F11.4,2X,2F9.3)")selsamsiz(j),table3(i,j,1:4),table3(i,j,5:6)
end do
end do
WRITE(399,FMT="(3X,A76)")line
WRITE(*,FMT="(3X,A76)")line
WRITE(*,*)
!++++++++++++++++++
! CALCULATE AND PRINT RUN TIME
!++++++++++++++++++
call  cpu_time(time2)
runtime=time2-time1
WRITE(*,*)' the runtime was ',runtime
stop
end program mcexp_verify
```

```
!    Last change:  AJT  15 Feb 2006    7:55 pm
program mcexp_rejectrates
!+++++++++++++++++++
! Programming begun July 12, 2005
! by  Andrew Tierman
! @@@                                                    @@@
! @@@ Revision: begin 11-11-2005 to accommodate alternatives 19 & 20 @@@
! @@@          archived copy retained                          @@@
!   (with adaptation of some code from overlap and
!    gravitystat programs, prior to topic of exponential testing)
!
! This program reads the four upper critical values of MADMASR
! stored in mcexptst01_outproc for four selected sample
! sizes - n = 4, 10, 20, and 50.
! The critical values are used to find
! Power (percent of rejections) for pseudo-
! random samples from a variety of non-exponential dist-
! tributions.
!
! OUTPUT:
! This program outputs Table Four: rejection rates for
! non-exponentially distributed data (Power)
!+++++++++++++++++++
!
!  EXTERNAL PROGRAMS REFERENCED
!
!+++++++++++++++++++
USE rangen
USE utils4tstexpdst
implicit none
!+++++++++++++++++++
!
!  TYPE DECLARATIONS
!
!+++++++++++++++++++
REAL(KIND=8)::time1,time2,runtime
REAL,ALLOCATABLE,DIMENSION(:)::xdata,xwork
REAL::xdata_mean,xdata_var,realtenthou
REAL::zero,one,two,six,rand1(1),rand2(1)
REAL::lambda,deci_lambda,sigma,nu,omega,oneoverlambda,mu,rho,tau,upsilon
REAL(KIND=8)::dseed
REAL::ytestat(100001),thetastars(100001),meann(100001),varn(100001)
REAL::pop15k_addhalves2mix(1500),pop15k_bernouli2mix(1500)   !@@@
REAL::alpha(4),rejectrate(4)
REAL::lamstar,thestar,medabsresid,madresid,maxabsdevfrmed
REAL::xbar,sd,skew,kurt,var
REAL::rgt_crit_4prcntls(4),empir_4prcntls(4)
REAL::statarray(8)
!REAL::table3(4,4,4) ! OP array: indices for parameter, sample size & alpha respectively
REAL::table4(20,4,8) ! Output Array for Table Four (Power of MADMASR statistic; distrib:sample
size:alpha)
REAL::table1a(4,4)  ! CV array for MADMASR: indices for selected sample size & alpha (from
Table1)
```

```
REAL::dist_pars(20, 3) ! indices for id, up to 3 parameters
INTEGER::dist_type(20) ! indices for id, type - 10 + 1 ('anomalous'@@@) distinct types of
distributions:
! DISTRIB TYPES: mix 2exp, mix exp&snd, gamma, Weibull, Gumbel, lognrml, Makeham, chisq,
unif, halfnrml
! @@@   also two 'anomalous' - 19) bernoulli mix and 20) half&half mix of a beta and a triangular
INTEGER::selsamsiz(4)
INTEGER::i,j,k,l,m,n,tenthou,distype,int_lambda,df,anomalous_unit
CHARACTER(LEN=35),ALLOCATABLE,DIMENSION(:)::nonexpdist_lg
CHARACTER(LEN=25),ALLOCATABLE,DIMENSION(:)::nonexpdist_st
CHARACTER(LEN=76)::line
CHARACTER(LEN=14)::ten5thmakehamn
CHARACTER(LEN=4)::txt
CHARACTER(LEN=2)::fn_n(4)
CHARACTER(LEN=20)::makehamdatafile(4),makehamdata
INTEGER::machine
CHARACTER(LEN=55)::machinepath(2)
CHARACTER(LEN=52)::datapath1    !for athelon XP 2800  - home
CHARACTER(LEN=36)::datapath2    !for pentium 4 2.6GHz - office
CHARACTER(LEN=42)::anomalous_datafile(10)    ! '1500Beta0.0-
0.0mix_add.5s_Tri.0.0.0mtb.txt'
!                                   '1500Beta0.0-0.0mix_Brnl.5_Tri.0.0.0mtb.txt'
!+++++++++++++++++
!
!  PROGRAM RUN TIMING;
!  VALUE INITIALIZATION;
!  ARRAY ALLOCATION
!
!++++++++++++++++++
call cpu_time(time1)
zero=0.0
one=1.0
two=2.0
six=6.0
anomalous_unit = 1100
tenthou = 100000  ! chgd to 10^5 repetitions, preserving "tenthou" for all program modules
realtenthou = REAL(tenthou)
dseed = 1.1111D9*COS(time1)
call CPU_TIME(time2)
dseed = dseed + 7777.0*SIN(time2)
line='___________________________________________________________________________
___'
selsamsiz(:) = (/4, 10, 20, 50/)
rgt_crit_4prcntls(:)=(/.90, .95, .975, .99/)
alpha(:) = one - rgt_crit_4prcntls(:)
m = 20               ! @@@ number of alternative distributions
ALLOCATE (nonexpdist_lg(m))
ALLOCATE (nonexpdist_st(m))
nonexpdist_lg(:) = (/'mixed exponential (0.5 & 4.0)       ','mixed exponential (1.0) & N(10,1)  ',&
&                  'gamma (shape = 1.6, scale = 0.625) ','gamma (shape = 4.0, scale = 0.25)  ',&
&                  'Weibull (shape = 0.5, scale = 0.5) ','Weibull (shape=1.5, scale=1.10773) ',& ! for
Weibull, see Bury pp 311-15
```

```
&                'Weibull (shape=2.0, scale=1.12838) ','Gumbel (location=.42278, scale=1.0)',& ! for
Gumbel, see Bury pp 267-71
&                'lognormal (shape =0.2, lnscale =0) ','lognormal (shape =1.0, lnscale =0) ',& !
Gumble=lrgst extr value dist
&                'lognormal (shape =2.0, lnscale =0) ','Makeham 1,1,1 (DLMF parameters)    ',& ! for
lgnml see Bury pp 154-57
&                'chi-square df = 1                 ','chi-square df = 2                 ',&
&                'chi-square df = 3                 ','chi-square df = 4                 ',&
&                'uniform [0, 1]                    ','half-normal                       ',&
&                '0.5*Beta(1.1,.4)+0.5*Tri(.2,.6,.7) ','BernouliMix Beta,1.1,.4&Tri.2,.6,.7'/)
nonexpdist_st(:) = (/'mixed: exp(0.5); exp(4.0)','mixed: exp(1); nrml(10,1)',&
&                'gamma (1.6, 0.625)       ','gamma (4.0, 0.25)       ',&
&                'Weibull (0.5, 0.5)       ','Weibull (1.5, 1.10773)   ',&
&                'Weibull (2.0, 1.128387)  ','Gumbel (0.42278, 1.0)    ',&
&                'lognormal (0.0, 0.2)     ','lognormal (0.0, 1.0)     ',&
&                'lognormal (0.0 ,2.0)     ','Makeham (1.0, 1.0 ,1.0)  ',&
&                'chi-square df 1          ','chi-square df2 [exp(2.0)]',&
&                'chi-square df 3          ','chi-square df 4          ',&
&                'uniform (0.0, 1.0)       ','half-normal (0.0, 1.0)   ',&
&                '.5*Bta1.1,.4+.5*Tri.2.6.7','BrnMx Bta1.1,.4&Tri.2.6.7'/)
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
++++++++++++++++++++
! DISTRIBUTION INFORMATION MATRIX  [Notation: see Bury, 1999]                 "9999" = none
!
! Type  Descr.      First Param   Second Param   Third Param    Used here
!                                            dist#    parameters
!  1    mix 2 exp's  theta1       theta2        none          1     0.5      4.0     9999
!               [scale]       [scale]
!..........................................................................................
!  2    mix exp&nrml theta        mean          std dev        2     1.0     10.0      1.0
!..........................................................................................
!  3    gamma       lambda       sigma         none          3     1.6      0.625   9999
!               [shape]       [scale]                        4     4.0      0.25    9999
!..........................................................................................
!  4    Weibull     lambda       sigma         none          5     0.5      0.5     9999
!               [shape]       [scale]                        6     1.5      1.10773 9999   NB for lambda=1,
!                                                            7     2.0      1.12838 9999   Weibull is Exp(sigma)
!..........................................................................................
!  5    Gumbel      mu           sigma         none          8     0.42278  1.0     9999
!               [location]    [scale]
!..........................................................................................
!  6    lognormal   mu           sigma         none          9     0.0      0.2     9999
!    {Bury, p. 154: exp(mu)=scale shape}                     10    0.0      1.0     9999
!               mean(ln(x))   sd(ln(x)) for rv,x             11    0.0      2.0     9999
!..........................................................................................
!  7    Makeham     xi           lambda        theta         12    1.0      1.0       1.0
!..........................................................................................
!  8    chi-square  df           none          none          13    1.0   9999       9999
!                                                            14    2.0   9999       9999  ! Chi-Sqr df=2 is exp w/
theta=2
!                                                            15    3.0   9999       9999
!                                                            16    4.0   9999       9999
!..........................................................................................
```

```
!  9    unif[0, 1]   a           b            none         17    0.0     1.0    9999
!.......................................................................................
! 10    half-normal  mu          sigma                                          1/2Nrml: see Gentle p 176
&
!                [for associated Normal]     none         18    0.0     1.0    9999   Hogg&Tanis 7th
pp 286-91
!.......................................................................................
! 11    anomalous mix of beta and triangular:                19    1500 variates half&half mix
beta(1.1,0.4)&Tri(.2,.6,.7.)
!      generated in MINITAB and stored in files              20    1500 variates Bernoulli mix
beta(1.1,0.4)&Tri(.2,.6,.7.)
!.......................................................................................
dist_type(:) = (/1, 2, 3, 3, 4, 4, 4, 5, 6, 6, 6, 7, 8, 8, 8, 8, 9, 10, 11, 11/) ! type of dist for each dist id
dist_pars(1,:) = (/0.5,  4.0, 9999.0/)      ! dist #1  mix of 2 exponentials
dist_pars(2,:) = (/1.0, 10.0, 1.0/)        ! dist #2  mix of exponential and normal
dist_pars(3,:) = (/1.6, 0.625, 9999.0/)     ! dist #3  gamma
dist_pars(4,:) = (/4.0, 0.25, 9999.0/)      ! dist #4  gamma  {=Erlang dist., for lambda a pos. integer;
Bury p. 214}
dist_pars(5,:) = (/0.5, 0.5, 9999.0/)       ! dist #5  Weibull
dist_pars(6,:) = (/1.5, 1.10773, 9999.0/)   ! dist #6  Weibull
dist_pars(7,:) = (/2.0, 1.12838, 9999.0/)   ! dist #7  Weibull
dist_pars(8,:) = (/0.42278, 1.0, 9999.0/)   ! dist #8  Gumbel
dist_pars(9,:) = (/0.0, 0.2, 9999.0/)       ! dist #9  lognormal
dist_pars(10,:) = (/0.0, 1.0, 9999.0/)      ! dist #10 lognormal
dist_pars(11,:) = (/0.0, 2.0, 9999.0/)      ! dist #11 lognormal
dist_pars(12,:) = (/1.0, 1.0, 1.0/)        ! dist #12 Makeham
dist_pars(13,:) = (/1.0, 9999.0, 9999.0/)   ! dist #13 Chi-square
dist_pars(14,:) = (/2.0, 9999.0, 9999.0/)   ! dist #14 Chi-square
dist_pars(15,:) = (/3.0, 9999.0, 9999.0/)   ! dist #15 Chi-square
dist_pars(16,:) = (/4.0, 9999.0, 9999.0/)   ! dist #16 Chi-square
dist_pars(17,:) = (/0.0, 1.0, 9999.0/)      ! dist #17 uniform
dist_pars(18,:) = (/0.0, 1.0, 9999.0/)      ! dist #18 half-normal
dist_pars(19,:) = (/5.0,9999.0,9999.0/)     ! dist #19 "parameter" id's anomalous datafile # for
"mix_add"
dist_pars(20,:) = (/6.0,9999.0,9999.0/)     ! dist #20 "parameter" id's anomalous datafile # for
"mix_Brnl"
anomalous_datafile(1:2) = (/'1500Beta0.1-0.3mix_add.5s_Tri.4.5.8mtb.txt','1500Beta0.1-
0.3mix_Brnl.5_Tri.4.5.8mtb.txt'/)
anomalous_datafile(3:4) = (/'1500Beta1.2-0.5mix_add.5s_Tri.3.4.6mtb.txt','1500Beta1.2-
0.5mix_Brnl.5_Tri.3.4.6mtb.txt'/)
anomalous_datafile(5:6) = (/'1500Beta1.1-0.4mix_add.5s_Tri.2.6.7mtb.txt','1500Beta1.1-
0.4mix_Brnl.5_Tri.2.6.7mtb.txt'/)
anomalous_datafile(7:8) = (/'                                       ','                                      '/)
anomalous_datafile(9:10) =(/'                                       ','                                      '/)
!++++++++++++++++++
!  Open output file for table 4, power of MADMASR statistic for alternative distributions
!++++++++++++++++++
OPEN(399,FILE='exptst_table4.txt',status='old')
!++++++++++++++++++
!
!  OPEN FILE AND READ CRITICAL
!  VALUES FOR SELECTED SAMPLE SIZES,
!  n = 4, 10, 20, 50, FROM TABLE 1
```

```
!
!++++++++++++++++++
OPEN(200,FILE='exptst_4by4cvarray1a.txt',status='old')  ! subscripts are for n and rt-tld-%ile
do i = 1,4
READ(200,FMT="(4F15.12)")table1a(i,:)
END do
!+++++++++++++++++
!
!  Open files for anomalously distributed data, N = 1500
!
!+++++++++++++++++
i = INT( dist_pars(19,1) )     ! "parameter" actually identifies data file
j = INT( dist_pars(20,1) )     ! "parameter" actually identifies data file
k = anomalous_unit + i
OPEN(k,FILE=anomalous_datafile(i),status='old')
READ(k,FMT="(F8.6)")pop15k_addhalves2mix    ! mix Beta and Triangular by adding halves of
variates
k = anomalous_unit + j
OPEN(k,FILE=anomalous_datafile(j),status='old')
READ(k,FMT="(F8.6)")pop15k_bernouli2mix     ! mix Beta and Triangular using Bernouli, p = .5 to
select variate
!++++++++++++++++++
!
!  Machine Identification and Makeham Data File Locations
!  machine 1 is an Athelon FX 2.083GHz system owned by A. Tierman, located at home
!  machine 2 is a Pentium 4, 2.6GHz system owned by SVSU, located in SW-337
!  The machine switch (value = 1 or 2 respectively) will permit a single command to
!  control the path for accessing the data files, with pseudorandom samples for four
!  selected sample sizes, producing using Dataplot (NIST), from the Makeham
!  (or Gompertz-Makeham) distribution.
!
!++++++++++++++++++! ******************* SET THIS SWITCH APPROPRIATELY PRIOR TO
COMPILE AND RUN
machine = 2        ! for greater convenience, this switch can be set automatically after a prompt
txt = '.txt'        ! for keyboard input during execution
ten5thmakehamn = 'ten5thmakehamn'
fn_n(:) = (/'04','10','20','50'/)
makehamdatafile(:) = ten5thmakehamn//fn_n(:)//txt
machinepath(:) = (/'c:\ATsFileStruct\AJTatWSUdissert\Makeham15f12july05\   ',&
&               'c:\ATonELF90\ATatWSU_TE_MakehamData\                '/)
datapath1 = 'c:\ATsFileStruct\AJTatWSUdissert\Makeham15f12july05\'
datapath2 = 'c:\ATonELF90\ATatWSU_TE_MakehamData\'
do i = 1,4      ! TWO DISTINCT PATH VARIABLES ARE NEEDED SINCE CHARACTER ARRAY
ELEMENTS MUST HAVE EQUAL LENGTH
makehamdata = makehamdatafile(i)
if (machine .eq. 1) OPEN(i,FILE=datapath1//makehamdata,STATUS='old')
if (machine .eq. 2) OPEN(i,FILE=datapath2//makehamdata,STATUS='old')
end do
!++++++++++++++++++
! P R I N C I P A L   L O O P S  for  M O N T E   C A R L O
!
!++++++++++++++++++
! L O O P   for  D I S T R I B U T I O N s
```

```
!+++++++++++++++++++
do i = 1, m          ! ECTO
!+++++++++++++++++++
! L O O P   for  S A M P L E   S I Z E S
!+++++++++++++++++++
do j = 1, 4          ! MESO
!+++++++++++++++++++
n = selsamsiz(j)
ALLOCATE (xdata(n), xwork(n))
meann(:) = zero
varn(:) = zero
!+++++++++++++++++++
! L O O P  for  R E P L I C A T I O N S
!+++++++++++++++++++
do k = 1, tenthou     ! ENDO
!+++++++++++++++++++
!+++++++++++++++++++
! ROUTING FOR DISTRIBUTIONS BY IF STRUCTURE, ON VARIABLE dist_type_pars(k dists;
10 types; <=3 parameters)
! DISTRIB TYPES: mix 2exp, mix exp&snd, gamma, Weibull, Gumbel, lognrml, Makeham, chisq,
unif, halfnrml
!+++++++++++++++++++
distype = dist_type(i)
if (distype .EQ. 1) then                              !dist type 1  mix 2 exps
   do l = 1, n
      call uni1(dseed, 1, rand1)
      if (rand1(1) .lt. .5) then
         call exp1(dseed, 1, dist_pars(i,1), rand2)
      else
         call exp1(dseed, 1, dist_pars(i,2), rand2)
      end if
      xdata(l) = rand2(1)
   end do
else if (distype .EQ. 2) then                         !dist type 2  mix exp & nrml
   do l = 1, n
      call uni1(dseed, 1, rand1)
      if (rand1(1) .lt. .5) then
         call exp1(dseed, 1, dist_pars(i,1), rand2)
      else
         call normb1(dseed, 1, rand2)
         rand2(1) = rand2(1)*dist_pars(i,3) + dist_pars(i,2)
      end if
      xdata(l) = rand2(1)
   end do
else if (distype .EQ. 3) then                         !dist type 3  gamma
   lambda = dist_pars(i,1)                             !lambda = shape param.
     int_lambda = INT(FLOOR(lambda))
     sigma = dist_pars(i,2)                            !sigma = scale param.
     if (lambda .EQ. int_lambda) then
        call erl1(dseed, n, int_lambda, sigma, xwork, xdata)
     else
     !for noninteger lambda, use the Cheng/Feast           NOT USED: Bury, pp 211-212 which
requires
```

```
     ! (1979) algorithm "for generating gamma              simulating Beta variates, using Gamma
     ! random variates when the shape parameter is           function; beginning of code not
used:
     ! greater than 1"   See Gentle pl 179 and cf Bury         !deci_lambda = lambda - int_lambda
        rho = lambda - one                             !call uni1(dseed, 1, rand1)
        tau = (six*lambda)**(-1)                        ! nu = -sigma*LOG(rand1)
gam:      do l = 1, n
1000         call uni1(dseed, 1, rand1)
          call uni1(dseed, 1, rand2)
          nu = (lambda - tau)*rand1(1)/(rho*rand2(1))
          omega = 2*(rand2(1) - one)/rho + nu + (one/nu)
          if (omega .le. 2.0) then
             xdata(l) = rho*nu*sigma
             CYCLE gam                                  !
          else
             upsilon = two*LOG(rand2(1))/rho - LOG(nu) + nu  !
             IF(upsilon .LE. one) then                    !
                xdata(l) = rho*nu*sigma                    !
                CYCLE gam                                 !
             END if
             GO TO 1000
          end if
        end do gam
     end if
else if (distype .EQ. 4) then                          !dist type 4  Weibull
     oneoverlambda = one/dist_pars(i, 1)                   ! see Bury, p. 315
     sigma = dist_pars(i, 2)
     call uni1(dseed, n, xdata)
     do l = 1, n
        xdata(l) = sigma*(LOG(one/xdata(l)))**oneoverlambda
     end do
else if (distype .EQ. 5) then                          !dist type 5  Gumbel
     mu = dist_pars(i, 1)                            ! see Bury, p. 271
     sigma = dist_pars(i, 2)
     call uni1(dseed, n, xdata)
     do l = 1, n
        xdata(l) = mu - sigma*LOG(LOG(one/xdata(l)))
     end do
else if (distype .EQ. 6) then                          !dist type 6  lognormal
     mu = dist_pars(i, 1)
     sigma = dist_pars(i, 2)
     call lnor1(dseed, n, mu, sigma, xdata)
else if (distype .EQ. 7) then                          !dist type 7  Makeham
     do l = 1, n                                    ! see Makeham Appendix; Dataplot
        READ(j,FMT="(f15.12)")xdata(l)
     end do
else if (distype .EQ. 8) then
     df = INT(dist_pars(i, 1))                        !dist type 8  Chi-square
     CALL chisq1(dseed, n, df, xwork, xdata)              ! note that chi-sq df=2 is exp theta=2
else if (distype .EQ. 9) then                          !dist type 9  uniform
     CALL uni1(dseed, n, xdata)
else if (distype .EQ. 10)then                          !dist type 10 half-normal
     mu = dist_pars(i, 1)
```

```
        sigma = dist_pars(i, 2)
        call normb1(dseed, n, xdata)
        xdata(:) = mu + sigma*xdata(:)
        xdata(:) = ABS(xdata(:))
else if (distype .eq. 11) then                    !dist type 11 anomalous data dist's
        if (i .eq. 19) then
            CALL randtafroara(dseed,n,1500,pop15k_addhalves2mix,xdata)
        else if (i .EQ. 20) then
            CALL randtafroara(dseed,n,1500,pop15k_bernouli2mix,xdata)
        end if
        !DIAGNOSTIC PRINT
        ! WRITE(*,*)xdata
else
        WRITE(*,*)' I N V A L I D   T Y P E'
end if
!+++++++++++++++++++
!  PROCESSING OF Single PSEUDORANDOM SAMPLE, dist #i, sample size(j) = n,  for
replication #k of 'tenthou' = 10^5
!  EVALUATE TEST STATISTIC, YTESTAT(k) [=MADMASR] FOR THIS SAMPLE, along with
associated statistics, including
!  the estimated exponential parameter, thestar, under the null hypothesis of exponentiality
!+++++++++++++++++++
call expon_star(n,xdata,lamstar,thestar,medabsresid,madresid,maxabsdevfrmed)
ytestat(k) = maxabsdevfrmed
thetastars(k) = thestar
!+++++++++++++++++++
!  Collect sample means and variances for average of each
!+++++++++++++++++++
call mean_and_var(n,xdata,xdata_mean,xdata_var)
!if (xdata_var .gt. 10**6) then
!   WRITE(*,*)xdata(:),xdata_var
!end if
meann(k) = xdata_mean
varn(k) = xdata_var
!+++++++++++++++++++
end do              ! ENDO  next k - replication
!WRITE(*,*)' Maxval of varn = ', maxval(varn)    DIAGNOSTIC FOR EXCESSIVE VARIANCE
!+++++++++++++++++++
! PROCESSING OF SET OF 'TENTHOU' = 10^5 REPLICATIONS OF YTESTAT(k)
[=MADMASR] and associated statistics
! for samples of given size, n, from given distribution #i
!+++++++++++++++++++
call four_stats(tenthou,thetastars,xbar,sd,skew,kurt)
statarray(1:4) = (/xbar, sd, skew, kurt/)
WRITE(*,*)nonexpdist_lg(i)
!WRITE(*,*)' tru_theta = ',tru_theta,'   mean thetastar = ',xbar
!WRITE(*,*)' mean estimated exponential parameter for all replications = ',xbar,' std dev = ',sd
call four_stats(tenthou,ytestat,xbar,sd,skew,kurt)
statarray(5:8) = (/xbar, sd, skew, kurt/)
call fastsort(tenthou,ytestat)
call upr_rej_rates(ytestat, tenthou, table1a(j,:), rejectrate)
table4(i,j,1:4) = rejectrate(:)
table4(i,j,5:6) = statarray(1:2)
```

```
table4(i,j,7) = SUM(meann)/realtenthou
table4(i,j,8) = SUM(varn)/realtenthou
!if (table4(i,j,8) .gt. 10**6) then                    DIAGNOSTIC FOR EXCESSIVE VARIANCE
!    WRITE(*,*)' avg var over 10^6 for dist & size ',i,j,table4(i,j,8)
!    stop
!end if
DEALLOCATE(xdata, xwork)
!++++++++++++++++++
end do              ! MESO  next j - sample size
!++++++++++++++++++
!++++++++++++++++++
end do              ! ECTO  next i - true sampled theta
!++++++++++++++++++
!++++++++++++++++++
!
!       P R I N T   T A B L E 4
!
!++++++++++++++++++
WRITE(*,*)
WRITE(399,FMT="(36X,'TABLE 4')")
WRITE(*,FMT="(36X,'TABLE 4')")
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(1X,A76)")line
WRITE(*,FMT="(1X,A76)")line
WRITE(399,*)
WRITE(*,*)
!         working Title of TABLE FOUR:   REJECTION RATES FOR ALTERNATIVE
DISTRIBUTIONS, for
!             distribution of the test statistic, MADMASR,
!              computed for 100,000 replications each, with random sampling for four sample
!                 sizes each for four exponential distributions, theta = ...
!
WRITE(399,FMT="(1X,'REJECTION RATES, on 100,000 replications with simulated random
sampling from')")
WRITE(399,FMT="(1X,I2,' alternative distributions respectively, each with sample sizes of 4,
10,')")m
WRITE(399,FMT="(1X,'20, and 50, for right-tailed alpha values of .10, .05, .025, and .01. The
mean')")
WRITE(399,FMT="(1X,'and std. dev. of the estimated exponential parameter and the average
sample')")
WRITE(399,FMT="(1X,'mean and sample variance for all replications are also shown for each
case.')")
!          SCREEN PRINT
WRITE(*,FMT="(1X,'REJECTION RATES, on 100,000 replications with simulated random
sampling from')")
WRITE(*,FMT="(1X,I2,' alternative distributions respectively, each with sample sizes of 4,
10,')")m
WRITE(*,FMT="(1X,'20, and 50, for right-tailed alpha values of .10, .05, .025, and .01. The
mean')")
WRITE(*,FMT="(1X,'and std. dev. of the estimated exponential parameter and the average
sample')")
```

```
WRITE(*,FMT="(1X,'mean and sample variance for all replications are also shown for each
case.')")
!
WRITE(399,FMT="(1X,A76)")line
WRITE(*,FMT="(1X,A76)")line
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(1X,'alpha',F6.3,3F9.3,9X,'theta-star',6X,'all samples')")alpha(:)
WRITE(*,FMT="(1X,'alpha',F6.3,3F9.3,9X,'theta-star',6X,'all samples')")alpha(:)
WRITE(399,FMT="(1X,'n',45X,'(est''d param.)',4X,'avg.    avg.')")
WRITE(*,FMT="(1X,'n',45X,'(est''d param.)',4X,'avg.    avg.')")
WRITE(399,FMT="(26X,'distribution',8X,'mean',3X,'std.dev.',4X,'mean    var.')")
WRITE(*,FMT="(26X,'distribution',8X,'mean',3X,'std.dev.',4X,'mean     var.')")
WRITE(399,FMT="(1X,A76)")line
WRITE(*,FMT="(1X,A76)")line
!        BODY OF TABLE 4
do i=1,m
WRITE(399,*)
WRITE(*,*)
WRITE(399,FMT="(26X,A25)")nonexpdist_st(i)
WRITE(*,FMT="(26X,A25)")nonexpdist_st(i)
WRITE(399,*)
WRITE(*,*)
do j=1,4
WRITE(399,FMT="(1X,I2,1X,4F9.4,1X,2F9.3,F10.3,F9.3)")selsamsiz(j),table4(i,j,1:4),table4(i,j,5:8
)
WRITE(*,FMT="(1X,I2,1X,4F9.4,1x,2F9.3,F10.3,F9.3)")selsamsiz(j),table4(i,j,1:4),table4(i,j,5:8)
end do
end do
WRITE(399,FMT="(4X,A70)")line
WRITE(*,FMT="(4X,A70)")line
WRITE(*,*)
!++++++++++++++++++
! CALCULATE AND PRINT RUN TIME
!++++++++++++++++++
call  cpu_time(time2)
runtime=time2-time1
WRITE(*,*)' the runtime was ',runtime
stop
!
end program mcexp_rejectrates
```

```
!    Last change:  AJT  15 Feb 2006    7:59 pm
program mcexptst_pwr
!""""""""""""""""""""""""""""""""
!"""Program for power analysis
!"""of several tests for the
!"""exponential distribution
!"""outputs Table 5
!""""""""""""""""""""""""""""""""
!"""by Andrew Tierman
!"""August 2005
! @@@                                       @@@
! @@@ Revision: begin 12-01-2005 to accommodate alternatives 19 & 20 @@@
! @@@         archived copy retained                          @@@
!""""""""""""""""""""""""""""""""
USE rangen
USE utils4tstexpdst
implicit none
!    Critical Value Table (test (1-5), sample size (n=4,10,20,50), alpha (.10,.05.,.01) )
REAL::cvtable(5,4,3)
!    Power Table  (test (1-5), sample size (1-4), signif. level (1-3), Alternative Distribution (18) )
!    for keys to identification of tests and distributions, see output formatting
REAL::pwrtable(5,4,3,20)    !@@@
REAL(KIND=8)::dseed,time1,time2,runtime
REAL,ALLOCATABLE,DIMENSION(:)::xdata
REAL,ALLOCATABLE,DIMENSION(:)::ksstat,cvmstat,swstat,ginistat,klcstat
REAL::pop15k_addhalves2mix(1500),pop15k_bernouli2mix(1500)   !@@@
!
REAL::theta,alpha(3),cv(3),rates(3)
INTEGER::iterations,iterno,i,j,k,n,m,sampsize(4),wndosize(4),rttlcritindx(3),machine
INTEGER::anomalous_unit,anom_data_set ! @@@
!
CHARACTER(LEN=14)::ten5thmakehamn
CHARACTER(LEN=4)::txt
CHARACTER(LEN=2)::fn_n(4)
CHARACTER(LEN=20)::makehamdatafile(4),makehamdata
CHARACTER(LEN=45)::machinepath(2)
CHARACTER(LEN=38)::datapath1    !for athelon XP 2800  - home
CHARACTER(LEN=36)::datapath2    !for pentium 4 2.6GHz - office
CHARACTER(LEN=35)::nonexpdist(20)
CHARACTER(LEN=4)::test_4plcname(5)
CHARACTER(LEN=42)::anomalous_datafile(10)    ! '1500Beta0.0-
0.0mix_add.5s_Tri.0.0.0mtb.txt' @@@
!                                    '1500Beta0.0-0.0mix_Brnl.5_Tri.0.0.0mtb.txt' @@@
!""""""""""""""""""""""""""""""""""""
!"""""INITIALIZATION""""""""""
!""""""""""""""""""""""""""""""""""""
call cpu_time(time1)
anomalous_unit = 1100    !@@@
nonexpdist(:) = (/'mixed exponential (0.5 & 4.0)      ','mixed exponential (1.0) & N(10,1)  ',&
&              'gamma (shape = 1.6, scale = 0.625) ','gamma (shape = 4.0, scale = 0.25)  ',&
&              'Weibull (shape = 0.5, scale = 0.5) ','Weibull (shape=1.5, scale=1.10773) ',& ! for
Weibull, see Bury pp 311-15
```

```
&               'Weibull (shape=2.0, scale=1.12838) ','Gumbel (location=.42278, scale=1.0)',& ! for
Gumbel, see Bury pp 267-71
&               'lognormal (shape =0.2, lnscale =0) ','lognormal (shape =1.0, lnscale =0) ',& !
Gumble=lrgst extr value dist
&               'lognormal (shape =2.0, lnscale =0) ','Makeham 1,1,1 (DLMF parameters)    ',& ! for
lgnml see Bury pp 154-57
&               'chi-square df = 1               ','chi-square df = 2               ',&
&               'chi-square df = 3               ','chi-square df = 4               ',&
&               'uniform [0, 1]                  ','half-normal                     ',& ! @@@Anomalous Dist's
&               '.5*Beta(1.1,0.4) + .5*Tri(.2,.6,.7)','BernouliMix Beta1.1,.4&Tri.2,.6,.7 '/) ! @@@## 19
& 22
!
test_4plcname(:) = (/' ks ',' cvm',' sw ','gini',' klc'/)
!
dseed = 1.2345D9*COS(time1)
iterations = 100000   !  for final run, use 100,000; use less for developmental runs
allocate
(ksstat(iterations),cvmstat(iterations),swstat(iterations),ginistat(iterations),klcstat(iterations))
theta = 1.0
!
call cpu_time(time2)
dseed = REAL(FLOOR(dseed + 6789.0*SIN(time2)),8)
!
sampsize(:) = (/4, 10, 20, 50/)
wndosize(:) = (/3, 3, 4, 5/)        !  Window size is required for the K-L, Correa test
alpha(:) = (/.10, .05, .01/)
rttlcritindx(:) = floor((1.0 - alpha(:))*iterations + 0.0)
!    initialize filenames for anomalous data [one set of halves-mix/Bernouli-mix will be selected for
opening] @@@
!    note that datafile(1:2)  were informally approved Nov. 30, 2005  and are preferred here.
anomalous_datafile(1:2) = (/'1500Beta0.1-0.3mix_add.5s_Tri.4.5.8mtb.txt','1500Beta0.1-
0.3mix_Brnl.5_Tri.4.5.8mtb.txt'/)
anomalous_datafile(3:4) = (/'1500Beta1.2-0.5mix_add.5s_Tri.3.4.6mtb.txt','1500Beta1.2-
0.5mix_Brnl.5_Tri.3.4.6mtb.txt'/)
anomalous_datafile(5:6) = (/'1500Beta1.1-0.4mix_add.5s_Tri.2.6.7mtb.txt','1500Beta1.1-
0.4mix_Brnl.5_Tri.2.6.7mtb.txt'/)
anomalous_datafile(7:8) = (/'                                        ','                                        '/)
anomalous_datafile(9:10) =(/'                                        ','                                        '/)
!
OPEN(499,FILE='exptst_table5.txt',status='old')      ! output file for power study
!''''''''''''''''''''''''''''''''''
!'''''''P R O C E D U R E S'''''''
!''''''''''''''''''''''''''''''''''
!''''''''     Print Output Header and Keys for tests(5) and alternative distributions(18)
! ''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''''
WRITE(*,FMT="(' The tests performed are based on the following statistics, as applied in the
given sources:')")
WRITE(499,FMT="(' The tests performed are the following:')")
WRITE(*,FMT="(5X,A4,' = Lilliefors-Kolmogorov-Smirnov statistic (Lilliefors,
1969)')")test_4plcname(1)
WRITE(499,FMT="(5X,A4,' = Lilliefors-Kolmogorov-Smirnov statistic (Lilliefors,
1969)')")test_4plcname(1)
```

```
WRITE(*,FMT="(5X,A4,' = Cramer-von Mises statistic with K transformation (Seshadri, Csorgo, &
Stephens, 1969)')")&
     &test_4plcname(2)
WRITE(499,FMT="(5X,A4,' = Cramer-von Mises statistic with K transformation (Seshadri,
Csorgo, & Stephens, 1969)')")&
     &test_4plcname(2)
WRITE(*,FMT="(5X,A4,' = Shapiro-Wilk statistic (Metz, Haccou, & Meelis,
1994)')")test_4plcname(3)
WRITE(499,FMT="(5X,A4,' = Shapiro-Wilk statistic (Metz, Haccou, & Meelis,
1994)')")test_4plcname(3)
WRITE(*,FMT="(5X,A4,' = Gini statistic (Gail & Gastwirth, 1978)')")test_4plcname(4)
WRITE(499,FMT="(5X,A4,' = Gini statistic (Gail & Gastwirth, 1978)')")test_4plcname(4)
WRITE(*,FMT="(5X,A4,' = Kullback-Liebler information statistic [except for Gumbel data] (Choi,
Kim, & Song, 2004)')")&
     &test_4plcname(5)
WRITE(499,FMT="(5X,A4,' = Kullback-Liebler information statistic [except for Gumbel data]
(Choi, Kim, & Song, 2004)')")&
     &test_4plcname(5)
WRITE(*,*)
WRITE(499,*)
!
!"""""     Obtain Critical Values for the five tests  SAMPLE SIZE / ALPHA / TEST
!"""""""""""""""""""""""""""""""""""""""""""
WRITE(*,FMT="(' CRITICAL VALUES FOR FIVE TESTS UNDER NULL HYPOTHESIS OF
EXPONENTIALITY')")
WRITE(499,FMT="(' CRITICAL VALUES FOR FIVE TESTS UNDER NULL HYPOTHESIS OF
EXPONENTIALITY')")
WRITE(*,*)
WRITE(499,*)
WRITE(*,FMT="(3(11X,'alpha = ',f5.2,13X))")alpha
WRITE(499,FMT="(3(11X,'alpha = ',f5.2,13X))")alpha
WRITE(*,FMT="(1X,3(5(A4,1X)2X))")test_4plcname(:),test_4plcname(:),test_4plcname(:)
WRITE(499,FMT="(1X,3(5(A5,2X)2X))")test_4plcname(:),test_4plcname(:),test_4plcname(:)
do i = 1, 4                  !  Begin loop for procedure for one sample size
n = sampsize(i)
WRITE(*,*)
WRITE(499,*)
WRITE(*,FMT="(' sample size = ',I4)")n
WRITE(499,FMT="(' sample size = ',I4)")n
!WRITE(*,*)
!WRITE(499,*)
m = wndosize(i)
ALLOCATE(xdata(n))
do iterno = 1, iterations     !  Begin loop for procedure for one iteration
call exp1(dseed,n,theta,xdata)
call fastsort(n, xdata)
!            Obtain test statistics for one pseudo-random sample
ksstat(iterno)   =   KS(n, xdata)
cvmstat(iterno)  =  CVM(n, xdata)
swstat(iterno)   =   SW(n, xdata)
ginistat(iterno) = GINI(n, xdata)
klcstat(iterno)  =  KLC(n, m, xdata)
!
```

```
end do                    !! End loop for procedure for one iteration
!
call fastsort(iterations,ksstat)
call fastsort(iterations,cvmstat)
call fastsort(iterations,swstat)
call fastsort(iterations,ginistat)
call fastsort(iterations,klcstat)
!          Obtain critical values for tests
!          cvtable(test,sampsize,rttlcritindx) for 5 tests, 4 sample sizes, & 3 alpha values
cvtable(1,i,:) =   ksstat(rttlcritindx(:))
cvtable(2,i,:) =  cvmstat(rttlcritindx(:))
cvtable(3,i,:) =   swstat(rttlcritindx(:))
cvtable(4,i,:) = ginistat(rttlcritindx(:))
cvtable(5,i,:) =  klcstat(rttlcritindx(:))
!          diagnostic print
WRITE(*,FMT="(1X,3(5(F4.2,1X)2X))")cvtable(:,i,1),cvtable(:,i,2),cvtable(:,i,3)
WRITE(499,FMT="(1X,3(5(F5.3,2X)2X))")cvtable(:,i,1),cvtable(:,i,2),cvtable(:,i,3)
deallocate (xdata)
end do                    !! End loop for procedure for one sample size
!
!""""""""""""""""""""""""""""""""""""""""
!"""""""""""""""" POWER STUDY""""20 Alternative Distributions and Five Tests  @@@
!""""""""""""""""""""""""""""""""""""""""
!
WRITE(*,*)
WRITE(499,*)
WRITE(*,FMT="(' POWER STUDY FOR FIVE TESTS FOR EXPONENTIALITY WITH 20
ALTERNATIVE DISTRIBUTIONS')")
WRITE(499,FMT="(' POWER STUDY FOR FIVE TESTS FOR EXPONENTIALITY WITH 20
ALTERNATIVE DISTRIBUTIONS')")
WRITE(*,*)
WRITE(499,*)
WRITE(*,FMT="(3(11X,'alpha = ',f5.2,13X))")alpha
WRITE(499,FMT="(3(11X,'alpha = ',f5.2,13X))")alpha
WRITE(*,FMT="(1X,3(5(A4,1X)2X))")test_4plcname(:),test_4plcname(:),test_4plcname(:)
WRITE(499,FMT="(1X,3(5(A5,2X)2X))")test_4plcname(:),test_4plcname(:),test_4plcname(:)
!
!""""""""""""""""""""""""""""""""""""""""""
!  Machine Identification and Makeham Data File Locations
!  machine 1 is an Athelon FX 2.083GHz system owned by A. Tierman, located at home
!  machine 2 is a Pentium 4, 2.6GHz system owned by SVSU, located in SW-337
!  The machine switch (value = 1 or 2 respectively) will permit a single command to
!  control the path for accessing the data files, with pseudorandom samples for four
!  selected sample sizes, producing using Dataplot (NIST), from the Makeham
!  (or Gompertz-Makeham) distribution.
!""""""""""""""""""""""""""""""""""""""""""
!******************* SET THIS SWITCH APPROPRIATELY PRIOR TO COMPILE AND RUN
machine = 2        ! for greater (?) convenience, this switch can be set automatically after a
prompt
txt = '.txt'       ! for keyboard input during execution
ten5thmakehamn = 'ten5thmakehamn'
fn_n(:) = (/'04','10','20','50'/)
makehamdatafile(:) = ten5thmakehamn//fn_n(:)//txt
```

```
machinepath(:) = (/'c:\AJTatWSUdissert\Makeham15f12july05\
','c:\ATonELF90\ATatWSU_TE_MakehamData\        '/)
datapath1 = 'c:\AJTatWSUdissert\Makeham15f12july05\'
datapath2 = 'c:\ATonELF90\ATatWSU_TE_MakehamData\'
do i = 1,4      ! TWO DISTINCT PATH VARIABLES ARE NEEDED SINCE CHARACTER ARRAY
ELEMENTS MUST HAVE EQUAL LENGTH
makehamdata = makehamdatafile(i)
if (machine .eq. 1) OPEN(i,FILE=datapath1//makehamdata,STATUS='old')
if (machine .eq. 2) OPEN(i,FILE=datapath2//makehamdata,STATUS='old')
end do
! """""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
!  Open anomalous data files & read data into arrays [one set of halves-mix/Bernouli-mix will be
selected] @@@
!  note that datafile(1:2)  were informally approved Nov. 30, 2005  and are preferred here.
@@@
anom_data_set = 3
i = (anom_data_set - 1)*2 + 1
j = (anom_data_set - 1)*2 + 2
k = anomalous_unit + i
OPEN(k,FILE=anomalous_datafile(i),status='old')
READ(k,FMT="(F8.6)")pop15k_addhalves2mix    ! mix Beta and Triangular by adding halves of
variates
k = anomalous_unit + j
OPEN(k,FILE=anomalous_datafile(j),status='old')
READ(k,FMT="(F8.6)")pop15k_bernouli2mix     ! mix Beta and Triangular using Bernouli, p = .5 to
select variate
!""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
!
!                    LOOP FOR DISTRIBUTIONS
do i = 1, 20           !begin loop for alternative distributions  @@@
WRITE(*,*)
WRITE(499,*)
!
WRITE(*,FMT="(' Alt. Dist. = ',A35)")nonexpdist(i)
WRITE(499,FMT="(' Alt. Dist. = ',A35)")nonexpdist(i)
!                        LOOP FOR SAMPLE SIZES
    do j = 1, 4            !begin loop for sample sizes
        n = sampsize(j)
        WRITE(*,*)
        WRITE(499,*)
        WRITE(*,FMT="(' sample size = ',I4)")n
        WRITE(499,FMT="(' sample size = ',I4)")n
        !WRITE(*,*)
        !WRITE(499,*)
        m = wndosize(j)
        ALLOCATE(xdata(n))
!                            LOOP FOR REPLICATIONS
        do iterno = 1, iterations !begin loop for iterations
!
!
!       CALL SUBROUTINE FOR MONTE CARLO SAMPLING @@@@ SUBROUTINE:
ALTDIST(seed, dist no., sample size, sample,
!                                                    anomalous data arrays 1 and 2)
```

```
!       obtain one pseudorandom sample and compute test statistics
            call altdist(dseed,i,j,n,xdata,pop15k_addhalves2mix,pop15k_bernouli2mix)
            call fastsort(n, xdata)
            !           Obtain test statistics for one pseudo-random sample
            ksstat(iterno)   =   KS(n, xdata)
            cvmstat(iterno)  =  CVM(n, xdata)
            swstat(iterno)   =   SW(n, xdata)
            ginistat(iterno) = GINI(n, xdata)
            if (i .eq. 8) then
               klcstat(iterno) = -99
              else
               klcstat(iterno)  =  KLC(n, m, xdata)
            end if
         END do                  !end loop for replications
!    Obtain right tail rejection rates for tests

        cv(:) = cvtable(1,j,:)
        call rt3rejrates(iterations,ksstat,cv,rates)  !works for unsorted teststatistics
        pwrtable(1,j,:,i) = rates(:)                 !indices are for test, sampsize, sig. lev., distrib.
        cv(:) = cvtable(2,j,:)
        call rt3rejrates(iterations,cvmstat,cv,rates)
        pwrtable(2,j,:,i) = rates(:)
        cv(:) = cvtable(3,j,:)
        call rt3rejrates(iterations,swstat,cv,rates)
        pwrtable(3,j,:,i) = rates(:)
        cv(:) = cvtable(4,j,:)
        call rt3rejrates(iterations,ginistat,cv,rates)
        pwrtable(4,j,:,i) = rates(:)
        cv(:) = cvtable(5,j,:)
        call rt3rejrates(iterations,klcstat,cv,rates)
        pwrtable(5,j,:,i) = rates(:)
 !
 !      diagnostic print
        WRITE(*,FMT="(1X,3(5(F4.2,1X)2X))")pwrtable(:,j,1,i),pwrtable(:,j,2,i),pwrtable(:,j,3,i)
        WRITE(499,FMT="(1X,3(5(F5.3,2X)2X))")pwrtable(:,j,1,i),pwrtable(:,j,2,i),pwrtable(:,j,3,i)
        deallocate (xdata)
    end do                  !end loop for sample sizes

!
end do                  !end loop for alternative distributions

!
!""""PROGRAM"""""""""""
!"""""""""TIMING""""""""
call cpu_time(time2)
runtime = time2 - time1
WRITE(*,*)
WRITE(*,*)' The runtime was ',runtime
WRITE(499,*)
WRITE(499,*)' The runtime was ',runtime
stop
!"""SUBPROGRAMS"""""""""""""""
contains
```

```
!"""functions to compute test statistics
!""""""""""""""""""""""""
!""""Kolmogorov-Smirnov Test Statistic   data must be sorted first
!
!""""""""""""""""""""""""
function KS(nn,xx)
INTEGER,INTENT(IN)::nn
REAL,INTENT(IN)::xx(nn)
REAL::KS
INTEGER::i
REAL::real_n,one,diff,diffhi,f0,f1,cdfdatum,thetahat
thetahat = SUM(xx)/nn
f0 = 0.0
diffhi = 0.0
one = 1.0
real_n = REAL(nn)
do i = 1, nn
f1 = i/real_n
cdfdatum = one - EXP(-(xx(i)/thetahat))
diff = MAX(ABS(f0 - cdfdatum),ABS(f1 - cdfdatum))
IF(diff .GT. diffhi) diffhi = diff
f0 = f1
end do
KS = diffhi
return
end function KS
!""""""""""""""""""""""""""""
!""""Cramer-von Mises Test Statistic       data must be sorted first
!
!""""""""""""""""""""""""""""
function CVM(n, x)
INTEGER,INTENT(IN)::n
REAL,INTENT(IN)::x(n)
REAL::y(0:n),z(n-2),s,d(n)
REAL::empir_cdf(n-2),CVM,diff(n-2)
INTEGER::i,m
!  ROUTINE FOR K TRANSFORMATION (from x to z)
y(0) = 0.0
y(1:n) = x(1:n)
d(1:n) = (/(i, i = n, 1, -1)/)*(y(1:n) - y(0:n-1))
s = SUM(d(2:n))  !  See Seshadri Csorgo & Stephens, 1969, page 503
m = n-2
do i = 1, m
    z(i) = SUM(d(2:i+1))/s
end do
!  C-vM statistic
empir_cdf(:) = (2*(/(i, i = 1, m)/) - 1)/(2.0*n)
diff = z - empir_cdf
CVM =1.0/(12.0*m) + DOT_PRODUCT(diff,diff)
return
end function CVM
!""""""""""""""""""""""""""""
!""""Shapiro-Wilk Test Statistic       data must be sorted first
```

```
!
!""""""""""""""""""""""""
function SW(nn, xx)
INTEGER,INTENT(IN)::nn
REAL,INTENT(IN)::xx(nn)
REAL::SW,xbar_xone,xbar,xvar
call mean_and_var(nn,xx,xbar,xvar)
xbar_xone = xbar - xx(1)
SW = nn*xbar_xone**2/(xvar*(nn - 1)**2)
return
end function SW
!""""""""""""""""""""""""
!"""Gini Test Statistic      data must be sorted first
!
!""""""""""""""""""""""""
function GINI(nn, xx)        !   See Gail and Gastwirth, eqn. 2.1 p. 351
INTEGER,INTENT(IN)::nn
REAL,INTENT(IN)::xx(nn)
REAL::GINI
INTEGER::i,nmo
INTEGER::ii(nn),jj(nn)
REAL::gnum(nn),gdenom
nmo = nn-1
ii(1:nmo)=(/(i,i=1,nmo)/)
jj(1:nmo)=nn-ii(1:nmo)
gnum(1:nmo)=ii(1:nmo)*jj(1:nmo)*(xx(2:nn)-xx(1:nmo))
gdenom = nmo*SUM(xx)
GINI = SUM(gnum)/gdenom
return
end function GINI
!"""""""""""""""""""""""""
!"""Kullback-Liebler Information Test Statistic   data must be sorted first
!
!       Correa's Entropy Estimator;
!       Choi, Kim & Song's K-L goodness of fit statistic for exponentiality
!"""""""""""""""""""""""""
function KLC(nn, mm, xx)        !  mm -- window provisions needed before entry, based on n
INTEGER,INTENT(IN)::nn, mm
REAL,INTENT(IN)::xx(nn)
REAL::KLC,CMN
INTEGER::i,j,jlo,jhi,twmmpluson,mmi,imm
REAL::one,xxbar,xx1,xxn
REAL::corxbar(nn),cmn_i(nn)
REAL,ALLOCATABLE,DIMENSION(:)::xjay,xdiff,cnum,cdenom
one = 1.0
twmmpluson = 2*mm + 1
xxbar = SUM(xx)/nn
jlo = 1 - mm
jhi = nn + mm
j = nn - 1
xx1=xx(1)
xxn=xx(nn)
ALLOCATE (xjay(jlo:jhi))
```

```
xjay(jlo:1) = xx1
xjay(2:j)   = xx(2:j)
xjay(nn:jhi)= xxn
!
do i = 1, nn
   mmi=i-mm
   imm=i+mm
   corxbar(i) = SUM(xjay(mmi:imm))/twmmpluson
   ALLOCATE(xdiff(mmi:imm),cnum(mmi:imm),cdenom(mmi:imm))
   xdiff(mmi:imm) = xjay(mmi:imm) - corxbar(i)
   cnum(mmi:imm) = xdiff(mmi:imm)*((/(j, j = mmi,imm)/) - i )
   cdenom(mmi:imm) = xdiff(mmi:imm)**2
   cmn_i(i) = - LOG(SUM(cnum)/(nn*SUM(cdenom)))/nn
   DEALLOCATE(xdiff,cnum,cdenom)
end do
CMN = SUM(cmn_i)
!
KLC = EXP(CMN)/EXP(LOG(xxbar) + one)
DEALLOCATE(xjay)
return
end function KLC
!
!
!
!""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
!      SUBROUTINE FOR
!      Monte Carlo Sampling - ALternative Distribution Selection Procedure -
!      revision for anomalous distributions 19 & 20 begun Dec. 1, 2005
!""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
!
subroutine altdist(dseed,i,j,n,xdata,anompop_hlvsmix,anompop_brnlmix)
!  Obtain pseudo-random sample xdata, size n, from the relevant alternative distribution, distrib.
no. = i
!   The sample size index, j, is needed to access the Makeham data file
USE rangen
USE utils4tstexpdst
implicit none
REAL(KIND=8),INTENT(IN OUT)::dseed
INTEGER,INTENT(IN)::i,j,n
REAL,INTENT(IN)::anompop_hlvsmix(1500),anompop_brnlmix(1500)
REAL,INTENT(OUT)::xdata(n)
!
REAL::dist_pars(20,3)     ! indices for dist. no. and up to three parameters
INTEGER::dist_type(20)    ! there are 11 distinct types of alternative distrib's:
!                       mix 2exp, mix exp&snd, gamma, Weibull, Gumbel,
!                       lognrml, Makeham, chisq, unif, halfnrml, anomalous mixture
INTEGER::l,int_lambda,df
REAL::rand1(1),rand2(1)
REAL::lambda,sigma,xwork(n),rho,one,tau,six,nu,omega,upsilon,two,oneoverlambda,mu
!""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
!""""""INITIALIZATION""""""including description of alternative distributions
!""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""""
one = 1.0
```

```
two = 2.0
six = 6.0
!
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
+++++++++++++++++++
! DISTRIBUTION INFORMATION MATRIX  [Notation: see Bury, 1999]                "9999" = none
!
! Type  Descr.      First Param   Second Param   Third Param    Used here
!                                              dist#    parameters              [E(X);VAR(X)]
!  1    mix 2 exp's  theta1       theta2       none          1     0.5      4.0     9999     [2.25, 11.1875]
!                [scale]      [scale]
!..............................................................................................
!  2    mix exp&nrml theta        mean         std dev        2    1.0     10.0      1.0     [5.5, 21.25]
!..............................................................................................
!  3    gamma       lambda       sigma         none          3     1.6      0.625   9999     [1, .625]
!                [shape]      [scale]                      4     4.0     0.25    9999     [1, .25]
!..............................................................................................
!  4    Weibull     lambda       sigma         none          5     0.5      0.5     9999     [1, 5]
! NB for lambda=1,    [shape]      [scale]                  6     1.5      1.10773 9999     [1, 1]
! Weibull is Exp(sigma)                                    7     2.0      1.12838 9999    ~ [.99997, .46097]
!..............................................................................................
!  5    Gumbel      mu           sigma         none          8     0.42278  1.0     9999     [1, ~1.64493]
!                [location]   [scale]
!..............................................................................................
!  6    lognormal   mu           sigma         none          9     0.0      0.2     9999    ~[1.02020,
.04248]
!    {Bury, p. 154: exp(mu)=scale shape}                    10     0.0      1.0     9999    ~[1.64872,
4.67077]
!                mean(ln(x))   sd(ln(x)) for rv,x          11     0.0      2.0     9999    ~[7.38906,
2926.35984]
!..............................................................................................
!  7    Makeham     xi           lambda        theta        12     1.0      1.0      1.0 ~ ~[.404, .112]
!..............................................................................................
!  8    chi-square  df           none          none         13     1.0   9999      9999     [1, 2]
! NB for df = 2,                                           14     2.0   9999      9999     [2, 4]
! Chi-Sq is exponential                                    15     3.0   9999      9999     [3, 6]
! w/ theta = 2                                             16     4.0   9999      9999     [4, 8]
!..............................................................................................
!  9    unif[0, 1]  a            b             none         17     0.0      1.0     9999     [.5, .0833..]
!..............................................................................................
! 10    half-normal  mu          sigma                                             1/2Nrml: see Gentle p 176
&
!                                                             Hogg&Tanis 7th pp 286-91
!                [for associated Normal]     none          18     0.0      1.0     9999    ~[.79788, .36338]
= (2/pi)^.5, 1-2/pi
!..............................................................................................
! 11    anomalous mix of beta and triangular:              19     1500 variates Bernoulli mix
beta(.1,.3)&Tri(.4,.5,.8.)Prf'd
!       generated in MINITAB and stored in files           20     1500 variates half&half mix
beta(.1,.3)&Tri(.4,.5,.8.)Prf'd
!..............................................................................................
dist_type(:) = (/1, 2, 3, 3, 4, 4, 4, 5, 6, 6, 6, 7, 8, 8, 8, 8, 9, 10, 11, 11/) ! type of dist for each dist id
dist_pars(1,:) = (/0.5,  4.0, 9999.0/)      ! dist #1  mix of 2 exponentials
```

```
dist_pars(2,:) = (/1.0, 10.0, 1.0/)        ! dist #2  mix of exponential and normal
dist_pars(3,:) = (/1.6, 0.625, 9999.0/)     ! dist #3  gamma
dist_pars(4,:) = (/4.0, 0.25, 9999.0/)      ! dist #4  gamma  {=Erlang dist., for lambda a pos. integer;
Bury p. 214}
dist_pars(5,:) = (/0.5, 0.5, 9999.0/)       ! dist #5  Weibull
dist_pars(6,:) = (/1.5, 1.10773, 9999.0/)   ! dist #6  Weibull
dist_pars(7,:) = (/2.0, 1.12838, 9999.0/)   ! dist #7  Weibull
dist_pars(8,:) = (/0.42278, 1.0, 9999.0/)   ! dist #8  Gumbel
dist_pars(9,:) = (/0.0, 0.2, 9999.0/)       ! dist #9  lognormal
dist_pars(10,:) = (/0.0, 1.0, 9999.0/)      ! dist #10 lognormal
dist_pars(11,:) = (/0.0, 2.0, 9999.0/)      ! dist #11 lognormal
dist_pars(12,:) = (/1.0, 1.0, 1.0/)         ! dist #12 Makeham
dist_pars(13,:) = (/1.0, 9999.0, 9999.0/)   ! dist #13 Chi-square
dist_pars(14,:) = (/2.0, 9999.0, 9999.0/)   ! dist #14 Chi-square
dist_pars(15,:) = (/3.0, 9999.0, 9999.0/)   ! dist #15 Chi-square
dist_pars(16,:) = (/4.0, 9999.0, 9999.0/)   ! dist #16 Chi-square
dist_pars(17,:) = (/0.0, 1.0, 9999.0/)      ! dist #17 uniform
dist_pars(18,:) = (/0.0, 1.0, 9999.0/)      ! dist #18 half-normal
dist_pars(19,:) = (/1.0,9999.0,9999.0/)     ! dist #19 anomalous datafile # for "mix_add" datafile
dist_pars(20,:) = (/2.0,9999.0,9999.0/)     ! dist #20 anomalous datafile # for "mix_bern" datafile
!
select case (dist_type(i))
!          !dist type 1  mix of 2 exps
   case (1)

    do l = 1, n
        call uni1(dseed, 1, rand1)
        if (rand1(1) .lt. .5) then
            call exp1(dseed, 1, dist_pars(i,1), rand2)
        else
            call exp1(dseed, 1, dist_pars(i,2), rand2)
        end if
        xdata(l) = rand2(1)
    end do
!          !dist type 2  mix exp & nrml
   CASE(2)
    do l = 1, n
        call uni1(dseed, 1, rand1)
        if (rand1(1) .lt. .5) then
            call exp1(dseed, 1, dist_pars(i,1), rand2)
        else
            call normb1(dseed, 1, rand2)
            rand2(1) = rand2(1)*dist_pars(i,3) + dist_pars(i,2)
        end if
        xdata(l) = rand2(1)
    end do
!            !dist type 3  gamma
   CASE(3)
    lambda = dist_pars(i,1)                          !lambda = shape param.
      int_lambda = INT(FLOOR(lambda))
      sigma = dist_pars(i,2)                         !sigma = scale param.
      if (lambda .EQ. int_lambda) then
          call erl1(dseed, n, int_lambda, sigma, xwork, xdata)
```

```
       else
       !for noninteger lambda, use the Cheng/Feast           NOT USED: Bury, pp 211-212 which
requires
       ! (1979) algorithm "for generating gamma               simulating Beta variates, using Gamma
       ! random variates when the shape parameter is            function; beginning of code not
used:
       ! greater than 1"   See Gentle pl 179 and cf Bury         !deci_lambda = lambda - int_lambda
           rho = lambda - one                                !call uni1(dseed, 1, rand1)
           tau = (six*lambda)**(-1)                           ! nu = -sigma*LOG(rand1)
gam:        do l = 1, n
1000            call uni1(dseed, 1, rand1)
            call uni1(dseed, 1, rand2)
            nu = (lambda - tau)*rand1(1)/(rho*rand2(1))
            omega = 2*(rand2(1) - one)/rho + nu + (one/nu)
            if (omega .le. 2.0) then
                xdata(l) = rho*nu*sigma
                CYCLE gam                               !
            else
                upsilon = two*LOG(rand2(1))/rho - LOG(nu) + nu  !
                IF(upsilon .LE. one) then                  !
                    xdata(l) = rho*nu*sigma                    !
                    CYCLE gam                               !
                END if
                GO TO 1000
            end if
        end do gam
       end if
!          !dist type 4  Weibull    see Bury, p. 315
      CASE(4)
       oneoverlambda = one/dist_pars(i, 1)
       sigma = dist_pars(i, 2)
       call uni1(dseed, n, xdata)
       do l = 1, n
           xdata(l) = sigma*(LOG(one/xdata(l)))**oneoverlambda
       end do
!          !dist type 5  Gumbel          see Bury, p. 271
      CASE(5)
       mu = dist_pars(i, 1)
       sigma = dist_pars(i, 2)
       call uni1(dseed, n, xdata)
       do l = 1, n
           xdata(l) = mu - sigma*LOG(LOG(one/xdata(l)))
       end do
!           !dist type 6  lognormal
      CASE(6)
       mu = dist_pars(i, 1)
       sigma = dist_pars(i, 2)
       call lnor1(dseed, n, mu, sigma, xdata)
!           !dist type 7  Makeham
      CASE(7)
!!       j = int(LOG(real(n))/LOG(2.5))
       do l = 1, n                                   ! see Makeham discussion; Dataplot
           READ(j,FMT="(f15.12)")xdata(l)
```

```
        end do
!           !dist type 8  Chi-square                      ! note that chi-sq df=2 is exp theta=2
       CASE(8)
        df = INT(dist_pars(i, 1))
        CALL chisq1(dseed, n, df, xwork, xdata)
!            !dist type 9  uniform
       CASE(9)
        CALL uni1(dseed, n, xdata)
!            !dist type 10 half-normal
       CASE(10)
        mu = dist_pars(i, 1)
        sigma = dist_pars(i, 2)
        call normb1(dseed, n, xdata)
        xdata(:) = mu + sigma*xdata(:)
        xdata(:) = ABS(xdata(:))
!            !dist type 11 anomalous data [finite pop., N = 1500]  @@@@
        CASE(11)
        if (i .eq. 19) then
            CALL randtafroara(dseed,n,1500,anompop_hlvsmix,xdata)
        else if (i .EQ. 20) then
            CALL randtafroara(dseed,n,1500,anompop_brnlmix,xdata)
        end if
!
       case default
end select
!
return
end subroutine altdist
!
!
!""""""""""""""""""""""""'
!""""'SUBROUTINE FOR RIGHT THREE REJECTION RATES
!
!""""""""""""""""""""""""'
subroutine rt3rejrates(nn,stats,cvs,rates)    !works for unsorted teststatistics
implicit none
INTEGER,INTENT(IN)::nn
REAL,INTENT(IN)::stats(nn),cvs(3)
REAL,INTENT(OUT)::rates(3)
!
REAL::rejcnt(3)
INTEGER::i
!
rejcnt(:) = 0.0
do i = 1, nn
      IF (stats(i) .GE. cvs(3)) then
          rejcnt(:) = rejcnt(:) + 1.0
        else if (stats(i) .ge. cvs(2)) then
          rejcnt(1:2) = rejcnt(1:2) + 1.0
        else if (stats(i) .ge. cvs(1)) then
          rejcnt(1) = rejcnt(1) + 1.0
      end if
end do
```

```
rates(:) = rejcnt(:)/nn
return
end subroutine rt3rejrates
!
!
end program mcexptst_pwr
```

```
!    Last change:  AT   13 Nov 2005    7:51 pm
module utils4tstexpdst
!*********************************************
!***   Utilities for Project to test       ***
!***   for Exponential Distribution        ***
!***   programmed by Andrew Tierman        ***
!***   coding  begun 1-23-2004             ***
!***   Last Revised Nov. 11, 2005          ***
!*********************************************
implicit none
INTEGER::switch
PUBLIC::switch
contains
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::                                           ::::::
!:::::   SUBROUTINE TO TAKE A RANDOM SAMPLE      ::::::
!:::::   OF GIVEN SIZE FROM A LARGER ARRAY OF    ::::::
!:::::   GIVEN SIZE  added 11 - 10 - 2005        ::::::
!:::::                                           ::::::
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine randtafroara(dseed,smpsiz,arasiz,array,randta)
USE rangen
implicit none
INTEGER,INTENT(IN)::smpsiz,arasiz
REAL,INTENT(IN)::array(arasiz)
REAL,INTENT(OUT)::randta(smpsiz)
REAL(KIND=8),INTENT(IN OUT)::dseed
REAL::unifdata(smpsiz),nines
INTEGER::intrdata(smpsiz)
!
nines = .99999
call uni1(dseed,smpsiz,unifdata)
intrdata(:) = FLOOR(arasiz*unifdata(:) + nines)
randta(:) = array(intrdata(:))
return
!
end subroutine randtafroara
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::                                           :::::
!:::::   SUBROUTINE TO  S O R T  A REAL,         :::::
!:::::                                           :::::
!:::::   ONE-DIMENSIONAL  A R R A Y, LOW TO HIGH   :::::
!:::::                                           :::::
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine sort_array(n,w)
INTEGER,INTENT(IN)::n
INTEGER::i,j
```

```
REAL, INTENT(IN OUT)::w(n)
REAL::save
do i=1,n-1
do j=i+1,n
SAVE=w(j)
if (w(i).gt.SAVE) then
w(j)=w(i)
w(i)=save
end if
end do
end do
return
end subroutine sort_array
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::                                          :::::
!:::::   SUBROUTINE TO  S O R T  A REAL,          :::::
!:::::                                          :::::
!:::::   ONE-DIMENSIONAL  A R R A Y, LOW TO HIGH   :::::
!:::::                                          :::::
!:::::      much faster than bubble sort above    :::::
!:::::                                          :::::
!:::::  See 8.2 "Heapsort", pp. 327-329 in         :::::
!:::::W. H. Press, S. A. Teukolsky, W. T. Vetterling,:::
!:::::and B. P. Flannery, Numerical Recipes in     :::::
!:::::Fortran 77: The Art of Scientific Computing, :::::
!:::::2nd ed., Vol. 1 of Fortran Numerical Recipes,:::::
!:::::Cambridge: Cambridge UniversityPress, 1992   :::::
!:::::                                          :::::
!::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine fastsort(n,w)
INTEGER,INTENT(IN)::n
REAL,INTENT(IN OUT)::w(n)
REAL::ww
INTEGER::h,ir,i,j
h=n/2+1
ir=n
100 if(h.gt.1)then
      h=h-1
      ww=w(h)
   else
      ww=w(ir)
      w(ir)=w(1)
      ir=ir-1
      IF(ir.eq.1)then
            w(1)=ww
            return
      end if
   end if
   i=h
   j=h+h
```

```
200 IF(j.le.ir)then
        IF(j.lt.ir)then
                IF(w(j).lt.w(j+1))j=j+1
        end if
        IF(ww.lt.w(j))then
                w(i)=w(j)
                i=j
                j=j+j
        else
                j=ir+1
        end if
    GO TO 200
    end if
    w(i)=ww
GO TO 100
end subroutine fastsort
!
!
!
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  SUBROUTINE TO COMPUTE SAMPLE MEAN & VARIANCE  :::::
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine mean_and_var(n,x,xbar,var)
implicit none
INTEGER, INTENT(IN)::n
REAL,INTENT(IN)::x(n)
REAL,INTENT(OUT)::xbar,var
!INTEGER::ii
!
!REAL::sumx,sumxsq,zero
!zero = 0.0
!sumx=zero
!sumxsq=zero
xbar = SUM(x)/n
var = (DOT_PRODUCT(x,x) - n*xbar**2) / (n - 1)
!do ii = 1,n
!sumx=sumx+x(ii)
!sumxsq=sumxsq+x(ii)**2
!end do
!xbar=sumx/n
!var=(sumxsq - sumx**2/n)/(n-1)
return
end subroutine mean_and_var
!
!
!
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  SUBROUTINE TO COMPUTE FOUR STATISTICS FOR A  ::::::
!::::::  REAL ARRAY - Mean, Standard Deviation,       ::::::
!::::::  Skewness, and Kurtosis                       ::::::
!::::::            [See Bury 1999 pp. 13-14]          ::::::
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine four_stats(n,x,smean,sstdev,sskewness,skurtosis)
```

```
implicit none
INTEGER, INTENT(IN)::n
REAL,INTENT(IN)::x(n)
REAL,INTENT(OUT)::smean,sstdev,sskewness,skurtosis
INTEGER::jj
REAL::x2(n),sumx2,sumx3,sumx4,sumsqrs,sumcubes,sumquads
!REAL::addupcubes,addupquads,holdcube
!REAL::moment_orig1,moment_orig2,moment_orig3,moment_orig4
!REAL::moment_mean2,moment_mean3,moment_mean4
REAL::zero,one,three_halves,two,three,four,six
REAL::n097,n098,n099,nreal,n101
!REAL::dev_from_smean(n)
zero=0.0
one=1.0
three_halves=1.5
two=2.0
three=3.0
four=4.0
six=6.0
nreal = REAL(n)
n097 = nreal - three
n098 = nreal - two
n099 = nreal - one
n101 = nreal + one
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
! The formulas for sample skewness and kurtosis used here  +
! appear in R.R. Sokal & F.J. Rohlf, Biometry: The princi- +
! ples and practice of statistics in biological research,  +
! 3d ed., NY: W.H.Freeman and Co., 1995, pp 114-115, and in+
! the help feature for Excel. Minitab and SPSS, as well as +
! Excel, appear to use these formulas, as determined by    +
! test results from small samples                          +
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
smean = SUM(x)/nreal
x2(:) = X(:)**2
sumx2 = DOT_PRODUCT(x,x)
sumx3 = DOT_PRODUCT(x,x2)
sumx4 = DOT_PRODUCT(x2,x2)
sumsqrs = sumx2 - nreal*smean**2
sumcubes = sumx3 - three*smean*sumx2 + two*nreal*smean**3
sumquads = sumx4 - four*smean*sumx3 + six*smean**2*sumx2 - three*nreal*smean**4
sstdev = SQRT(sumsqrs/n099)
sskewness = (nreal*sumcubes)/(n099*n098*sstdev**3)
skurtosis = (n101*nreal*sumquads)/(n099*n098*n097*sstdev**4) -
(three*n099*n099)/(n098*n097)
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!  FORMULAS for sample skewness and kurtosis are those     +
!  used in EXCEL (with documentation) and are consistent   +
!  with results for these statistics in MINITAB and SPSS   +
!  The formulas below derived from population moments are  +
!  archived but not used here.                             +
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!moment_orig3 = addupcubes/n
```

```
!moment_orig4 = addupquads/n
!moment_mean2 = moment_orig2 - moment_orig1**two
!moment_mean3 = moment_orig3 - three*moment_orig2*moment_orig1 +
two*moment_orig1**three
!moment_mean4 = moment_orig4 - four*moment_orig3*moment_orig1 +&
!&six*moment_orig2*moment_orig1**two - three*moment_orig1**four
!smean = moment_orig1
!sstdev = SQRT(moment_mean2)
!sskewness = moment_mean3/(moment_mean2**three_halves)
!skurtosis = moment_mean4/(moment_mean2**two)
return
end subroutine four_stats
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  SUBROUTINE TO FIND CRITICAL PERCENTILES OF THE
!::::::  EMPIRICAL, MONTE CARLO-DERIVED DISTRIBUTION OF
!::::::  TEST STATISTICS FROM SORTED ARRAY -
!::::::  THIS ROUTINE WILL FIND 4 LOWER PERCENTILES
!::::::  TRANSFERRED IN FROM CALLING PROGRAM
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine lft_tld_crt_val4(nn,ydata,crit_prcntl,empir_prcntl)
implicit none
INTEGER,INTENT(IN)::nn
INTEGER::i,j,low
REAL,INTENT(IN)::ydata(nn),crit_prcntl(4)
REAL,INTENT(OUT)::empir_prcntl(4)
REAL::half,one,two,real_index
half=0.5
one=1.0
two=2.0
!
!   references on computing percentiles--page 25, Zar, Biostatistical Analysis, 3rd
!   more explicit: Rosner, Fundamentals of Biostatistics, 5th, page 19, followed here.
!   j is the index of the percentile
do i = 1, 4
real_index=crit_prcntl(i)*(nn+1)
low=FLOOR(real_index)
if (low.lt.real_index) then
  empir_prcntl(i) = ydata(low+1)
else
  empir_prcntl(i) = (ydata(low)+ydata(low+1))/two
end if
end do
return
end subroutine lft_tld_crt_val4
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  SUBROUTINE TO FIND CRITICAL PERCENTILES OF THE
!::::::  EMPIRICAL, MONTE CARLO-DERIVED DISTRIBUTION OF
```

```
!::::::  TEST STATISTICS FROM SORTED ARRAY -
!::::::  THIS ROUTINE WILL FIND 4 UPPER PERCENTILES
!::::::  TRANSFERRED IN FROM CALLING PROGRAM
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine rgt_tld_crt_val4(nn,ydata,crit_prcntl,empir_prcntl)
implicit none
INTEGER,INTENT(IN)::nn
INTEGER::i,j,low
REAL,INTENT(IN)::ydata(nn),crit_prcntl(4)
REAL,INTENT(OUT)::empir_prcntl(4)
REAL::half,one,two,real_index
REAL::rgt_crit_prcntl(4)
half=0.5
one=1.0
two=2.0
!
!   references on computing percentiles--page 25, Zar, Biostatistical Analysis, 3rd
!   more explicit: Rosner, Fundamentals of Biostatistics, 5th, page 19, followed here.
!   j is the index of the percentile
do i = 1, 4
j=4-i+1
rgt_crit_prcntl(i)=one-crit_prcntl(j)
real_index=rgt_crit_prcntl(i)*(nn+1)
low=FLOOR(real_index)
if (low.lt.real_index) then
  empir_prcntl(i) = ydata(low+1)
else
  empir_prcntl(i) = (ydata(low)+ydata(low+1))/two
end if
end do
return
end subroutine rgt_tld_crt_val4
!
!
!
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  FUNCTION TO COMPUTE SUM OF SQUARES - 1VAR     :::::
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::
function sumofsqrs(n,x)
implicit none
INTEGER, INTENT(IN)::n
REAL, INTENT(IN)::x(n)
REAL::sumofsqrs
sumofsqrs = DOT_PRODUCT(x,x) - (SUM(x)**2)/n
return
end function sumofsqrs
!
!
!
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!::::::  FUNCTION TO COMPUTE SUM OF SQUARES - CrossProducts  :::::
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
function sumofsqrs_cp(n,u,v)
```

```
implicit none
INTEGER,INTENT(IN)::n
REAL,INTENT(IN)::u(n),v(n)
REAL::sumofsqrs_cp
sumofsqrs_cp = DOT_PRODUCT(u,v) - SUM(u)*SUM(v)/n
return
END function sumofsqrs_cp
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::      P R I N C I P A L   S U B R O U T I N E  FOR        ::
!:::::       T E S T I N G   E X P O N E N T I A L I T Y       ::
!:::::  SUBROUTINE TO RECEIVE VARIABLE ARRAY AND COMPARE          ::
!:::::    EXPONENTIAL CDF USING THE LEAST SQUARES LAMBDA METHOD   ::
!:::::      TO THE EMPIRICAL DISTRIBUTION                       ::
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine expon_star(n,x,lamstar,thestar,medabsresid,madresid,maxabsdevfrmed)
implicit none
INTEGER,INTENT(IN)::n
REAL,INTENT(IN)::x(n)
REAL,INTENT(OUT)::lamstar,thestar,medabsresid,madresid,maxabsdevfrmed
INTEGER::i,imedloc
REAL::zero,half,one,negone,two
REAL::xbar,var,s,log_arg,i_log,bld_numer,bld_denom,res_stddev
REAL::xsorted(n)
REAL::f_inv(n),resid(n),stdresid(n),absresid_lo2hi(n),absdevfrmed(n)
REAL::ss_x,ss_f,ss_fx,b1,b0,r,resmn,resvar
REAL::rmedloc,addup,addupsq
zero=0.0
half=0.5
one=1.0
negone=-one
two=2.0
bld_numer=zero
bld_denom=zero
xsorted=x
call fastsort(n,xsorted)
call mean_and_var(n,xsorted,xbar,var)
s=SQRT(var)
!     TASK 1.  CALCULATE LAMBDA STAR POSITIONING THETA STAR (THE
!              EXPONENTIAL PARAMETER) BETWEEN THE SAMPLE MEAN AND
!              SAMPLE STANDARD DEVIATION AT THE LOCATION WHICH
!              MINIMIZES THE SUM OF SQUARED DIFFERENCES BETWEEN
!              THE INPUT VECTOR AND THE INVERSE EXPONENTIAL CDF
!              AT THE RESPECTIVE QUANTILES OF THE EMPIRICAL
!              DISTRIBUTION OF THE DATA
!              Note: the theoretical mean and standard deviation
!              of an exponential distribution are both equal to
!              the distribution parameter [Miller&Miller, John E.
!              Freund's Mathematical Statistics, 1999; p. 209]
!
!
```

```
!   CORRECT THE FOLLOWING; MAY 12, 2005 - - - - - - -
!do i = 1,n
!log_arg=one-(two*i-one)/(two*n)
!i_log=LOG(log_arg)
!bld_numer=bld_numer+(xsorted(i)+xbar*i_log)*i_log
!bld_denom=bld_denom+i_log**2
!end do
!lamstar=bld_numer/((xbar-s)*bld_denom)
!++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 2.  CALCULATE THETA STAR, THE EXPONENTIAL PARAMETER
!
!thestar=xbar+lamstar*(s-xbar)
!  - - - - - - - REPLACING THE PRECEDING WITH THE FOLLOWING CODE:  (MAY 12, 2005)
do i = 1 , n
log_arg=one-(two*i-one)/(two*n)
i_log=LOG(log_arg)
bld_numer =  bld_numer + xsorted(i)*i_log
bld_denom =  bld_denom + i_log**2
end do
!
!  If statement determines case: whether sample mean is below or above sample std dev
!
!!if (xbar.LT.s) then
!!       lamstar = (xbar + bld_numer/bld_denom)/(xbar - s)
!!       thestar  = xbar + lamstar*(s - xbar)
!!       else
!!       lamstar  = (s + bld_numer/bld_denom)/(s - xbar)
!!       thestar  =  s + lamstar*(xbar - s)
!!end if
thestar = negone*bld_numer/bld_denom
lamstar = 99999  !dummy value for unused lamstar  AMENDMENT MAY 20, 2005
!!DIAGNOSTIC PRINT!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
!WRITE(*,FMT="(' n= ',I4,' xbar= ',f6.3,' s= ',f6.3,' theta*= ',f6.3)")n,xbar,s,thestar
!++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 3.  FILL AN ARRAY WITH INVERSE EXPONENTIAL CDF OF
!              QUANTILES OF THE EMPIRICAL DISTRIBUTION
!
do i=1,n
log_arg=one-(two*i-one)/(two*n)
i_log=LOG(log_arg)
f_inv(i)=negone*thestar*i_log
end do
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 4.  REGRESS INVERSE CDF ON INPUT ARRAY
!
ss_x=sumofsqrs(n,xsorted)
ss_f=sumofsqrs(n,f_inv)
ss_fx=sumofsqrs_cp(n,xsorted,f_inv)
b1=ss_fx/ss_x
r=ss_fx/(SQRT(ss_x*ss_f))
b0=SUM(f_inv)/n - b1*xbar
!++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 5.  GET ARRAY OF STANDARDIZED ABSOLUTE RESIDUALS
```

```
!          first, calculate array of residuals for the sample
do i=1,n
resid(i)=f_inv(i)-(b0+b1*xsorted(i))
END do
!        take Abs Val to get absolute residuals at this point - amended 4-20-05
absresid_lo2hi=ABS(resid)
!        get mean and std dev of abs resids for standardization - amended 4-20-05
addupsq=zero
do i=1,n
addupsq=addupsq+absresid_lo2hi(i)**2
end do
addup=SUM(absresid_lo2hi)
resmn=addup/n
resvar=(addupsq-n*resmn**2)/(n-one)
if (resvar .lt. zero) then
!     write (*,FMT="(E20.10E8)")' RESVAR IN UTILS IS LESS THAN ZERO, = ',resvar
      WRITE(*,*)' RESVAR IN UTILS IS LESS THAN ZERO, = ',resvar
      resvar = TINY(one)
      switch = 1
END if
res_stddev=SQRT(resvar)
!       standardize the absolute residuals - amended 4-20-05
stdresid=(absresid_lo2hi-resmn)/res_stddev
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 6.  SORT, AND COMPUTE MEDIAN OF, STANDARDIZED ABSOLUTE RESIDUAL
!              ARRAY AND THE MEDIAN ABSOLUTE DEVIATION FROM
!              THE MEDIAN OF THE ARRAY (the last called madresid
!              in this routine)  - amende 4-20-05
!
!-----------------------------------------------------------------
!resid=resid/thestar    WHY? removed 4-20-05
!-----------------------------------------------------------------
call fastsort(n,stdresid)   ! CORRECTED 4-20-05
rmedloc=(n+one)/two
imedloc=(n+1)/2
if (floor(rmedloc+half).EQ.imedloc) then          ! CORRECTED 4-20-05
    medabsresid=stdresid(imedloc)
    else
    medabsresid=(stdresid(INT(rmedloc-half))+stdresid(INT(rmedloc+half)))/two
end if
absdevfrmed=ABS(stdresid-medabsresid)           ! CORRECTED 4-20-05
call fastsort(n,absdevfrmed)
if (floor(rmedloc+half).EQ.imedloc) then         !CORRECTED 4-03-04
    madresid=absdevfrmed(imedloc)
    else
    madresid=(absdevfrmed(INT(rmedloc-half))+absdevfrmed(INT(rmedloc+half)))/two
end if
maxabsdevfrmed =  absdevfrmed(n)         !ADDED 4-5-04 with coordinated revisions for new
variable
!maxabsdevfrmed = maxabsdevfrmed/sqrt(REAL(n))   ! ADDED FOR TESTING, July 18, 2005,
after low rejection rates (table 4)
!the foregoing operation does not substantially change the low rejection rates, although there is
more dispersion in rates.
```

```
!  WHAT IF??? SEE FOLLOWING TRIAL REVISION JULY 19, 2005
!maxabsdevfrmed=stdresid(n)  !also tried: maxabsdevfrmed = medabsresid
return
end subroutine expon_star
!
!
!
!
!
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::: SUBROUTINE TO COMPUTE UPPER REJECTION          :::::
!::::::        (TYPE I ERROR) RATE,                :::::  Subroutine added 7-10-2005
!:::::: FOR RECEIVED SORTED TEST STATISTIC ARRAY AND    :::::
!:::::: ARRAY SIZE, AND ARRAY OF UPPER CRITICAL VALUES   :::::
!:::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
subroutine upr_rej_rates(test, n, cv, rejectrate)
implicit none
!......arguments.......
INTEGER, INTENT(IN)::n
REAL, INTENT(IN)::test(n)
REAL, INTENT(in)::cv(4)
REAL, INTENT(OUT)::rejectrate(4)
!......internal variables......
INTEGER::error_count(4), i, j, ione, inegone
ione = 1
inegone = -1
i = n
error_count(:) = 0
!......structure to accumulate errors within overlapping critical regions....
do j = 4, ione, inegone
        do WHILE ((test(i) .ge. cv(j)) .AND. (i .ge. ione))
        error_count(ione:j) = error_count(ione:j) + ione
        i = i + inegone
        end do
end do
rejectrate = REAL(error_count)/n
return
end subroutine upr_rej_rates
!
!
!
!
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
!:::::     P R I N C I P A L   S U B R O U T I N E  FOR       ::
!:::::     T E S T I N G   T W O - P A R A M E T E R         ::
!:::::         E X P O N E N T I A L I T Y                  ::
!:::::     2 par version: August 9-10, 2005                  ::
!:::::  SUBROUTINE TO RECEIVE VARIABLE ARRAY AND ESTIMATE        ::
!:::::  EXPONENTIAL CDF USING THE LEAST SQUARES LAMBDA METHOD     ::
!:::::  TO COMPARE WITH THE EMPIRICAL DISTRIBUTION -            ::
!:::::  generates test statistic, maxabsdevfrmed                ::
!::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::
```

```
subroutine
expon2_star(n,x,lamstar,mu_star,sigma_star,medabsresid,madresid,maxabsdevfrmed)
implicit none
INTEGER,INTENT(IN)::n
REAL,INTENT(IN)::x(n)
REAL,INTENT(OUT)::lamstar,medabsresid,madresid,maxabsdevfrmed,mu_star,sigma_star
INTEGER::i,imedloc
REAL::zero,half,one,two
REAL::xbar,var,res_stddev
REAL::xsorted(n)
REAL::f_inv(n),resid(n),stdresid(n),absresid_lo2hi(n),absdevfrmed(n)
REAL::ss_x,ss_f,ss_fx,b1,b0,r,resmn,resvar
REAL::rmedloc,addup,addupsq
REAL::two_en,logterm(n),lgtmsqrd(n),numerator,denominator,xstdev,sumlogterm,sumlgtmsqrd
REAL::lam1,lam2,aa1,bb1,aa2,bb2,ess_bar,bar_one,R1,R2,R3
zero=0.0
half=0.5
one=1.0
two=2.0
xsorted=x
call fastsort(n,xsorted)
call mean_and_var(n,xsorted,xbar,var)
!     TASK 1.  CALCULATE LAMBDA STAR POSITIONING THETA STAR based
!          on a linear combination of the sample minimum and
!          sample standard deviation, equal to the sample mean.
!          See Bury, 1999, on 2- parameter exponential distribution.
!          ---so that lambda star ----
!          MINIMIZES THE SUM OF SQUARED DIFFERENCES BETWEEN
!          THE INPUT VECTOR AND THE INVERSE EXPONENTIAL CDF
!          AT THE RESPECTIVE QUANTILES OF THE EMPIRICAL
!          DISTRIBUTION OF THE DATA
!          Note: the theoretical mean and standard deviation
!          of a one parameter exponential distribution are both equal to
!          the distribution parameter [Miller&Miller, John E.
!          Freund's Mathematical Statistics, 1999; p. 209]
!          The minimum statistic is an estimator of the location
!          parameter for the two-parameter exponential distribution
!
!          Initiated August 9, 2005.  A. Tierman
!++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!     TASK 2.  CALCULATE THETA STAR, THE EXPONENTIAL PARAMETER
!
xstdev = SQRT(var)
two_en = two*n
!!   array  logterm(n)
logterm(:) = LOG(one - (two*(/(i, i = 1, n)/) - one)/two_en)
!lgtmsqrd(:) = logterm(:)**2
!sumlogterm = SUM(logterm)
!sumlgtmsqrd = SUM(lgtmsqrd)
!numerator = n*xbar + (sumlogterm + xbar + xbar*sumlgtmsqrd/xstdev) / xstdev
!denominator = n + xsorted(1)*( one + sumlogterm + sumlgtmsqrd/xstdev) / xstdev
!lamstar = numerator / denominator
!mu_star = lamstar*xsorted(1)
```

```
!sigma_star = (xbar - lamstar*xsorted(1)) / xstdev
!
R1 = SUM(logterm)
R2 = DOT_PRODUCT(xsorted,logterm)
R3 = DOT_PRODUCT(logterm,logterm)
ess_bar = xstdev - xbar
bar_one = xbar - xsorted(1)
aa1 = R3/ess_bar**2
bb1 = (bar_one*(n*ess_bar + R1 - R1*R3) - R1*R2 + xsorted(1)*R1**2) / (n*R3 - R1**2)
aa2 = R1/(xstdev*ess_bar**2)
bb2 = (bar_one*(n*ess_bar + R1 - n*R3)-n*R2 + n*xsorted(1)*R1) / (R1**2 - xstdev*R3)
lam1 = aa1/bb1
lam2 = aa2/bb2
mu_star = xsorted(1) + lam1*ess_bar
sigma_star= bar_one - lam2*xstdev
lamstar = 1
!!!!mu_star = (n*xsorted(1)-xbar)/(n-1)      ! see Bury point-estimates for 2 parameter model, page
187
!!!!sigma_star = n*(xbar - xsorted(1))/(n-1) ! same.
!
!!DIAGNOSTIC PRINT!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
!WRITE(*,FMT="(' n= ',I4,' xbar= ',f6.3,' s= ',f6.3,' theta*= ',f6.3)")n,xbar,s,thestar
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 3.  FILL AN ARRAY WITH INVERSE EXPONENTIAL CDF OF
!             QUANTILES OF THE EMPIRICAL DISTRIBUTION
!
f_inv(:) = mu_star - sigma_star*logterm(:)
!++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 4.  REGRESS INVERSE CDF ON INPUT ARRAY
!
ss_x=sumofsqrs(n,xsorted)
ss_f=sumofsqrs(n,f_inv)
ss_fx=sumofsqrs_cp(n,xsorted,f_inv)
b1=ss_fx/ss_x
r=ss_fx/(SQRT(ss_x*ss_f))
b0=SUM(f_inv)/n - b1*xbar
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!      TASK 5.  GET ARRAY OF STANDARDIZED ABSOLUTE RESIDUALS
!          first, calculate array of residuals for the sample
do i=1,n
resid(i)=f_inv(i)-(b0+b1*xsorted(i))
END do
!         take Abs Val to get absolute residuals at this point - amended 4-20-05
absresid_lo2hi=ABS(resid)
!         get mean and std dev of abs resids for standardization - amended 4-20-05
addupsq=zero
!do i=1,n
!addupsq=addupsq+absresid_lo2hi(i)**2
!end do
addupsq = DOT_PRODUCT(absresid_lo2hi,absresid_lo2hi)
addup=SUM(absresid_lo2hi)
resmn=addup/n
resvar=(addupsq-n*resmn**2)/(n-one)
```

```
if (resvar .lt. zero) then
!    write (*,FMT="(E20.10E8)")' RESVAR IN UTILS IS LESS THAN ZERO, = ',resvar
     WRITE(*,*)' RESVAR IN UTILS IS LESS THAN ZERO, = ',resvar
     resvar = TINY(one)
     switch = 1
END if
res_stddev=SQRT(resvar)
!       standardize the absolute residuals - amended 4-20-05
stdresid=(absresid_lo2hi-resmn)/res_stddev
!+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
!     TASK 6.  SORT, AND COMPUTE MEDIAN OF, STANDARDIZED ABSOLUTE RESIDUAL
!              ARRAY AND THE MEDIAN ABSOLUTE DEVIATION FROM
!              THE MEDIAN OF THE ARRAY (the last called madresid
!              in this routine)  - amende 4-20-05
!
!------------------------------------------------------------------
!resid=resid/thestar    WHY? removed 4-20-05
!------------------------------------------------------------------
call fastsort(n,stdresid)   ! CORRECTED 4-20-05
rmedloc=(n+one)/two
imedloc=(n+1)/2
if (floor(rmedloc+half).EQ.imedloc) then          ! CORRECTED 4-20-05
    medabsresid=stdresid(imedloc)
    else
    medabsresid=(stdresid(INT(rmedloc-half))+stdresid(INT(rmedloc+half)))/two
end if
absdevfrmed=ABS(stdresid-medabsresid)           ! CORRECTED 4-20-05
call fastsort(n,absdevfrmed)
if (floor(rmedloc+half).EQ.imedloc) then         !CORRECTED 4-03-04
    madresid=absdevfrmed(imedloc)
    else
    madresid=(absdevfrmed(INT(rmedloc-half))+absdevfrmed(INT(rmedloc+half)))/two
end if
maxabsdevfrmed =  absdevfrmed(n)         !ADDED 4-5-04 with coordinated revisions for new
variable
!maxabsdevfrmed = maxabsdevfrmed/sqrt(REAL(n))   ! ADDED FOR TESTING, July 18, 2005,
after low rejection rates (table 4)
!the foregoing operation does not substantially change the low rejection rates, although there is
more dispersion in rates.
!  WHAT IF??? SEE FOLLOWING TRIAL REVISION JULY 19, 2005
!maxabsdevfrmed=stdresid(n)  !also tried: maxabsdevfrmed = medabsresid
return
end subroutine expon2_star
!
!
!
END module utils4tstexpdst
```

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ABSTRACT

TESTING FOR EXPONENTIALITY USING A TWO-MOMENT ESTIMATOR AND A MEDIAN-CENTERED DISTANCE STATISTIC

by

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August 2006

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Degree: Doctor of Philosophy

This is a Monte Carlo study of a new goodness-of-fit test for exponentiality, utilizing maximum absolute deviation from median deviation of the data from an estimated exponential distribution. The parameter estimation utilizes the fact that the mean and standard deviation are equal for a one-parameter exponential distribution to find a least-squares best fit parameter between the sample mean and sample standard deviation. 100,000 replications are used for each case in the computer simulations throughout. Critical values are computed for sample sizes between 3 and 100. Samples sized n = 4, 10, 20, and 50 are used for the remainder of the study. Nineteen selected non-exponential distributions are simulated for the power study. Five competing tests for exponentiality were compared: Lilliefors-Kolmogorov-Smirnov (Lilliefors, 1969); Cramer-von Mises with K transformation (Seshadri, Csorgo, and

Stephens, 1969); Shapiro-Wilk (Metz, Haccou, and Meelis, 1994); the Gini statistic (Gail and Gastwirth, 1978); and the Kullbach-Liebler discrimination statistic with Correa's entropy estimator (Choi, Kim, and Song, 2004). Each test was applied through computer simulations to verify critical values in the literature and to obtain rejection rates for the alternative distributions.

AUTOBIOGRAPHICAL STATEMENT

ANDREW JAY TIERMAN

Education:

Doctor of Philosophy, Evaluation and Research, quantitative concentration, Wayne State University (WSU), 2006

Master of Arts in Mathematics, Central Michigan University (CMU), 1989

Michigan Provisional Secondary Teaching Certificate: endorsements in mathematics, English, sociology, political science, and physics, Saginaw Valley State University (SVSU), 1988, 1989

Master of Liberal Studies in American Culture, University of Michigan; title of thesis: *Morality, Moral Cognition, And Social Structure: A Critical, Humanist Review*, 1987

Juris Doctor, Detroit College of Law; studies in History of the English Legal System, French Legal System, and International Law at the Universities of Exeter, England, and Windsor, Ontario, 1973

Bachelor of Arts in Mathematics, WSU, 1968

Employment:

Lecturer in Mathematical Sciences, SVSU, 1991-present: beginning algebra through college algebra and trigonometry; finite mathematics including business calculus; general statistics; and biostatistics

Adjunct Instructor, graduate teaching assistant, substitute teacher, summer youth program instructor, Upward Bound and adult education instructor: Delta College, CMU, SVSU, and Bay City, Buena Vista, Saginaw, and Saginaw Township School Districts, 1986-1991

Staff Attorney, Legal Services of Eastern Michigan (Saginaw), 1975-1985: served diverse population providing free counsel and representation in civil litigation and state and federal administrative proceedings, including domestic relations, housing, consumer protection, and public benefits law

VISTA Attorney, Nebraska Indian Inter-Tribal Development Corporation and Georgia Legal Services, 1973-1975: provided legal counsel and representation to members of three Native American Tribes in Nebraska and to indigent residents of Columbus, GA and surrounding rural area

Mathematical Analyst, Chrysler Corporation, Highland Park, 1969-1970: programming in FORTRAN and CDC 6400 assembler; applications in industrial design, engineering and manufacturing; implementation of software translating numerical control programs into data for automatic milling machines; in-house development of computer-aided design and manufacturing